Here is the kind of question we promise *not* to ask you very often:

> Arlene made a 75¢ purchase.
> She gave the clerk a dollar bill.
> How much change should she get?

That's an uninteresting question. All Arlene needs to know is that she should receive a quarter's worth of change.

In this book you are more likely to find questions like this:

> How many different combinations of pennies, nickels, and dimes make a quarter's worth of change?

You probably don't know the answer at once. Almost no one does. Almost anyone can answer the question if he will think about it, do a little planning, and carry out the plan using pencil and paper when necessary.

Someone might start a list of combinations such as:

> 5 nickels
> 1 dime, 1 nickel, 10 pennies
> 2 dimes and 5 pennies
> 1 nickel and 20 pennies

Such an approach could quickly lead him into trouble. How will he know whether or not he has listed *all* possible combinations?

Planning Ahead

Trouble can be avoided by working out a plan to list all possible combinations.

Here is an example of such a plan:

A. The combinations which include 2 dimes:

> 2 dimes, 1 nickel, 0 pennies
> 2 dimes, 0 nickels, 5 pennies

B. The combinations which include just 1 dime:

> 1 dime, 3 nickels,   0 pennies
> 1 dime, 2 nickels,   5 pennies
> 1 dime, 1 nickel,   10 pennies
> 1 dime, 0 nickels, 15 pennies

C. The combinations which include zero dimes:

> 0 dimes, 5 nickels,   0 pennies
> 0 dimes, 4 nickels,   5 pennies
> 0 dimes, 3 nickels, 10 pennies
> 0 dimes, 2 nickels, 15 pennies
> 0 dimes, 1 nickel,   20 pennies
> 0 dimes, 0 nickels, 25 pennies

There are exactly 12 different combinations of dimes, nickels, and pennies that make a quarter's worth of change.

Notice the following:

1. We thought about the problem.
2. We devised a plan to solve the problem.
3. We carried out the plan using paper and pencil when necessary.
4. We turned up new information which helped us solve the problem.
5. We can always use this plan to solve problems like this one.

1

Perhaps some of you are curious enough to raise new questions about the problem we solved.

Again, you don't know, but you can find out. Let's put all combinations in a table, and note the total number of coins in each combination:

D — number of dimes

N — number of nickels

P — number of pennies

C — number of coins

D	2	2	1								
N	1	0	3								
P	0	5	0								
C	3	7									

We've started the table. Please complete it.

Do any of the combinations have the same number of coins?

Perhaps you would like to think about two more questions:

You may be able to find the answers to the last 2 questions by simply studying the table of combinations of coins that are worth 25¢. You may decide to make a chart to help you answer each question.

The point we are trying to make is this: You can devise a plan which can help you to solve the problem for yourself.

You may wish to explore other problems such as:

The authors have never tried to solve this problem. If you do solve it, perhaps you will send us your solution.

Thumb through the book you have in your hand. It contains many questions you will want to solve on your own or with a little help from others.

THE SCIENCE FAIR PROJECT OF ALEC MARSON

It was truly an odd looking machine that Alec Marson entered in the Fairview School Science Fair. When the principal, Mr. Wilson, looked for Alec, he was gone. He had left town with his family.

"No one knows where they have gone," Mr. Wilson explained to the judges of the Science Fair. "In fact, no one knows where they came from."

The judges were puzzled by Alec's machine.

The machine had no covers and no screws to take out. It seemed as if it were made out of a single piece of strange metal.

What is it?

What does it do?

Should it be opened up to see what's inside?

An Experiment

"Let's see if anything happens when I write on the green screen that's marked, 'You write here,'" said one of the judges. "I suppose I could use that thing that looks like a pencil wired into the machine."

He wrote a 2 on the green screen. The other pencil wrote a 4 on the blue screen.

"Something happened; but I'm not sure I know anything more than I knew before."

"Let me try it," another judge asked. He pushed the ERASE key. The green and blue screens went blank. He wrote 0 on the green screen. The machine wrote 0 on the blue screen.

Another judge pushed the ERASE key, and then wrote 5 on the green screen.

The machine wrote 10 on the blue screen.

"That's interesting," Mr. Wilson said. "Let's keep a record of these experiments."
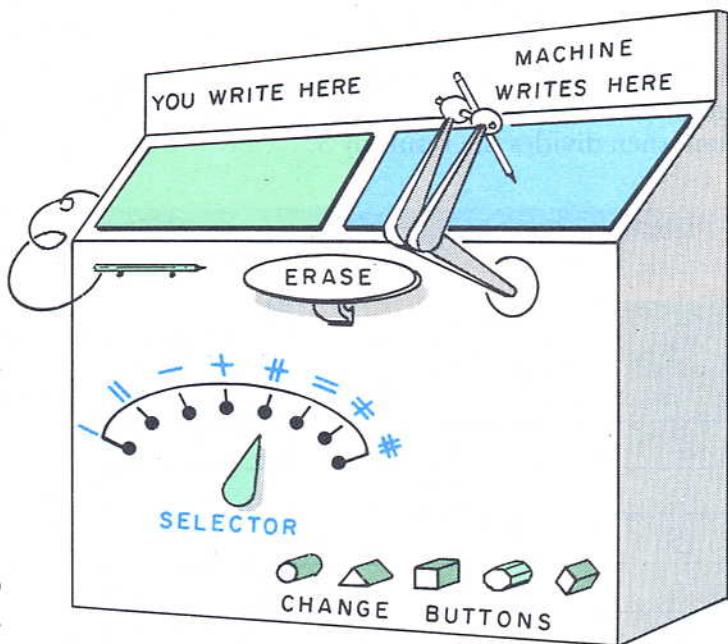
He drew a table on the board to keep track of the results. Here is a part of the table. Please fill in the missing numbers.

You write	2	0	5	1	7	25	
Machine writes	4	0	10	2			18

Judge Harris said, "I think the machine adds 5 to any number written; then doubles the result; and, finally, subtracts 10."

(*Please fill in the missing numbers in the table.*)

Machine does:

You write	add 5	double	subtract 10	Machine writes
2	7	14	4	4
0	5			0
5		20		10
1				2
7				
25				
				18
23				
	16			

3

ANOTHER SUGGESTION

Judge Thornton said, "Perhaps the machine multiplies the number you have written by 6, and then divides the result by 3."

(*Please fill in the missing numbers in the table.*)

You write	Machine's Work		Machine writes
	multiply by 6	divide by 3	
2			
0			
5			
1			
7			
25			

Mr. Wilson, the principal, suggested that perhaps the machine wasn't as complicated as the judges thought it was.

"Maybe it simply doubles whatever number you write."

Judge Jones offered another explanation.

"Maybe the machine adds 3, triples that sum, subtracts 8 from the product, and finally subtracts 1 more than the number you write."

You write	add 3	triple	subtract 8	subtract 1 more than the no. you write	Mach. writes
2	5	15	7	4	
0	3	9			
5	8				
1					
7					
25					
9					
	26				
		42			
			31		
		60		34	
	53				100

MORE SUGGESTIONS

Other suggestions were as follows:

a. Multiply by 10 and divide by 5.

b. Multiply by 3 and subtract the number you write.

c. Halve the number you write and multiply the result by 4.

d. Subtract 5, double the result, and then add 10. (Do you think the machine can subtract 5 from 0 or from 1 or 2 or 3 or 4?)

Mr. Wilson was a little impatient. "I'm sure," he said, "that Alec Marson could have built a machine that got the same results by an even more complicated sequence of operations.

"He might have had his machine add 2 to the number you write; multiply by 6, add 9, divide by 3, and subtract 7."

(*Please fill in the missing numbers in the table.*)

You write	2	0	5	1	7	3	4
add 2	4						
multiply by 6	24						
add 9	33						
divide by 3	11						
subtract 7	4						
Machine writes	4						

"However," Mr. Wilson continued, "Alec Marson was a very bright boy. He seldom used complicated methods to do something if he could find a way to do the same thing more simply.

"I'm convinced he built a machine that simply doubles any number you write on the green screen.

"I think that there is a lot more to Alec's machine than we think we know about yet. Let's see what else it can do."

Mr. Wilson explained that Alec had thought Cross Number Puzzles were fun. The judges had never heard of such puzzles.

"In one form," Mr. Wilson explained, "four numbers are written in a group of four boxes.

You add the numbers in each row and each column separately, and then add each pair of sums." Mr. Wilson illustrated by drawing on the green screen. To everyone's amazement, the machine started writing on the blue screen.

"Amazing!" the judges said in unison. Judge Thornton wanted to try a Cross Number Puzzle with larger numbers.

Mr. Wilson said, "Alec knew how to complete puzzles when the numbers were arranged differently. Perhaps the machine also knows how to complete such puzzles."

Judge Jones shook his head. "That boy certainly is unusual."

"Yes," replied Mr. Wilson, "he is, indeed. He also enjoyed working with Magic Squares. I wonder what would happen if I put a Magic Square on the green screen."

Mr. Wilson started a Magic Square. The machine completed it for him. Please complete this Magic Square.

The machine also wrote on the blue screen. Please finish this Magic Square.

Judge Thornton said, "I agree with Mr. Wilson. The machine is set to double everything. Let's call it a Doubling Machine."

"Not yet," Mr. Wilson cautioned. "I want to know what happens when we move the pointer along the dial marked Selector."

Mr. Wilson turned the pointer from # to #. He wrote 4 on the green screen. The machine wrote 16 on the blue screen.

"I think the machine is multiplying by 4," said Judge Thornton.

Mr. Wilson wrote 6 and the machine wrote 18.

Did the machine make a mistake?

Mr. Wilson made a table. Please fill in the missing numbers in the table.

A.

You write	4	6	12	9	13	24	25
Machine writes	16	18	24	21	25		

Mr. Wilson concluded that the machine did not make a mistake.

Mr. Thornton said, "The machine adds 6 to the number written, doubles the result and subtracts the number you write.

"Please fill in the missing numbers in my table."

B.

You write	4	6	12	9	13	24	25
Add 6.	10	12					
Double the sum.	20	24					
Subtract the no. you write.	16						
Machine writes	16	18	24				

Judge Jones said, "The machine doubles the number you write, adds 24 to the result, and then divides by 2.

"Please fill in the missing numbers in my table."

C.

You write	4	6	12	9	13	24	25
Double the no.	8						
Add 24.	32						
Divide by 2.							
Machine writes							

Judge Harris said, "Add 4, triple the result, and subtract twice the number you write."

"Please fill in the missing numbers in my table."

D.

You write	4	6	12	9	13	24	25
Add 4.	8	10					
Triple the sum.	24	30					
Subtract twice no. you write.		30 -12					
Machine writes		18					

Mr. Wilson was growing impatient again.

"No," he said. "Alec always tried to avoid complications. I'm sure that when the selector points to # , the machine adds 12 to whatever number you write.

Judge Harris admitted that they certainly had missed the easy way.

Let's try another Cross Number Puzzle.

E.

You write here.			Machine writes here.	
9	7		12	10
		50		
29				

Experimenting with the Selector

The judges moved the selector to each of the eight positions indicated.

Mr. Wilson kept track of the results, some of which are shown in the tables below.

Please fill in the missing numbers in each table.

Selector position

I

	a.	b.	c.	d.	e.	f.	g.	h.	i.	j.	k.	l.	m.
You write	1	2	5	14	10	9	0	8	50	75	106	500	610
Machine writes	3	5	11	29									

II

	a.	b.	c.	d.	e.	f.	g.	h.	i.	j.	k.	l.	m.
You write	0	1	2	3	5	7	50	6	44	72	110	314	350
Machine writes	3	6	9	12	18								

—

	a.	b.	c.	d.	e.	f.	g.	h.	i.	j.	k.	l.	m.
You write	10	2	7	1	11	3	8	5	15	25	31	200	320
Machine writes	19	3	13	1	21								

+

	a.	b.	c.	d.	e.	f.	g.	h.	i.	j.	k.	l.	m.
You write	3	1	5	2		7		9		20	25	307	515
Machine writes	18	6	30	12	24		36		48				

#

	a.	b.	c.	d.	e.	f.	g.	h.	i.	j.	k.	l.	m.
You write	0	7	1	4	19	36	49	28					199
Machine writes	0	14	2	8					200	250	702	322	

=

	a.	b.	c.	d.	e.	f.	g.	h.	i.	j.	k.	l.	m.
You write	2	1	5	10	4				8	100	111	220	213
Machine writes	9	5	21	41		25	37	13					

‡

	a.	b.	c.	d.	e.	f.	g.	h.	i.	j.	k.	l.	m.
You write	0	11	15	23	13		12						
Machine writes	0	44	60	92		84		56	100	72	88	368	496

#

	a.	b.	c.	d.	e.	f.	g.	h.	i.	j.	k.	l.	m.
You write	1	7	8	48						108	318	490	899
Machine writes	13	19	20	60	12	41	62	70	100				

When the pointer of Alec Marson's machine was at " ‡ ," Mr. Wilson tried another Cross Number Puzzle. He used some numbers which were larger than those he used in the previous puzzles.

Please complete the puzzles.

You write here.

	add ↓	add ↓	add ↓	add ↓
add →	100		7	
add →		70		
add →		90	16	

The machine writes here.

400		28	
1200			

The judges decided to adjourn until the next morning.

Mr. Wilson took Alec's machine home with him.

In his study at home, he pushed the triangular button. Two Cross Number Puzzle forms flashed on the green screen and one flashed on the blue screen. Mr. Wilson went to work. Finish Mr. Wilson's work for him.

A.

You write here.

20	8	
30	4	

$+$

9	10	
7	6	

Machine writes here.

29		
	10	

Here are some more puzzles started by Mr. Wilson.

The green forms appeared on the green screen. The blue forms on the blue screen. Please complete the puzzles.

B.

	13	
		36
71		100

$+$

	8	
		37
80		95

$=$

	21	
151		

C.

	7	
21		58

$+$

	25	
15		
23		59

$=$

25	18	

D.

		124
200		
	40	

$+$

300		351
	58	

$=$

	75	
300		
	149	

E.

45		
32	110	

$+$

29	125	
		58
57		

$=$

		549

8

"I wonder what will happen if I move the pointer over to ' II ,'" said Mr. Wilson. He left the triangular button pressed down.

Two Cross Number Puzzle forms appeared on the green screen, with a minus sign between them.

A.

		15
	15	
25		50

−

2		
	3	
6		10

=

	9	12
19		

B.

22		
		54
	47	

−

	7	10
9		

=

	1	
6		
		53

C.

	100	

−

		8
	8	12
10		

=

4	18	
26	32	

D.

	30	63
20		37

−

10	12	
11		

=

23		
	4	13

E.

300		342
800	91	

−

=

100	25	
300	27	

F.

	80	405
	106	581

−

90	7	
	36	

=

60		79
350	63	

"I wish I could see inside Alec's machine," said Mr. Wilson. "Oh well, let's see what else it'll do."

Mr. Wilson moved the selector pointer to " — ." Magic Squares appeared where Cross Number Puzzle forms had been.

A

			15
4	3		15
		1	15
2		6	15
15	15	15	15

+

B
33

		17
19	11	
	15	

=

A + B

	10		
28		4	48
		19	

C

16	2	
6	10	
		4

+

D

	25	

=

C + D

56		42
	35	49
28		

E

+

F

		16
	20	
24	4	

=

E + F

14	63	
	35	21
	7	

G

	7	
5	9	13
10		

+

H

	22	
		28
25	26	

=

G + H

	29	
35		27

Mr. Wilson moved the selector to " **+** ." "Ah-hah!" he said aloud, "now I can work on Cross Number Puzzle multiplication."

A.

100	30	4	
		5	385
100	40		
500		15	

×2 =

600	160		
		12	
1,000			

B.

100		3	
		6	

×5 =

500	450		995
	100		615
500			780
1,500	800		

C.

	40	5	
		8	
	120	15	735

×3 =

600			
		60	
900			

D.

200	20		
300	60		
200	60		
			856

×6 =

		54	
1,200	360	30	
4,200	840		

E.

1,000	70	8	
1,500	110	21	

×4 =

400		16	
	160		

11

Sale

Limit: 3 each to a customer

Magazines......... 4¢ ea.
Paperbacks .... 7¢ ea.
Hardback books ...16¢ ea.

Limit: 3 ea. to a customer

II

Specials

Pencils 5¢ ea.
Rulers 9¢ ea.
Notebooks 20¢ ea.

Limit: 3 ea. to a customer

III

Today Only!

Model Cars
Model Planes   7¢ ea.
Model Rockets   13¢ ea.
                   22¢ ea.

Limit: 3 ea. to a customer
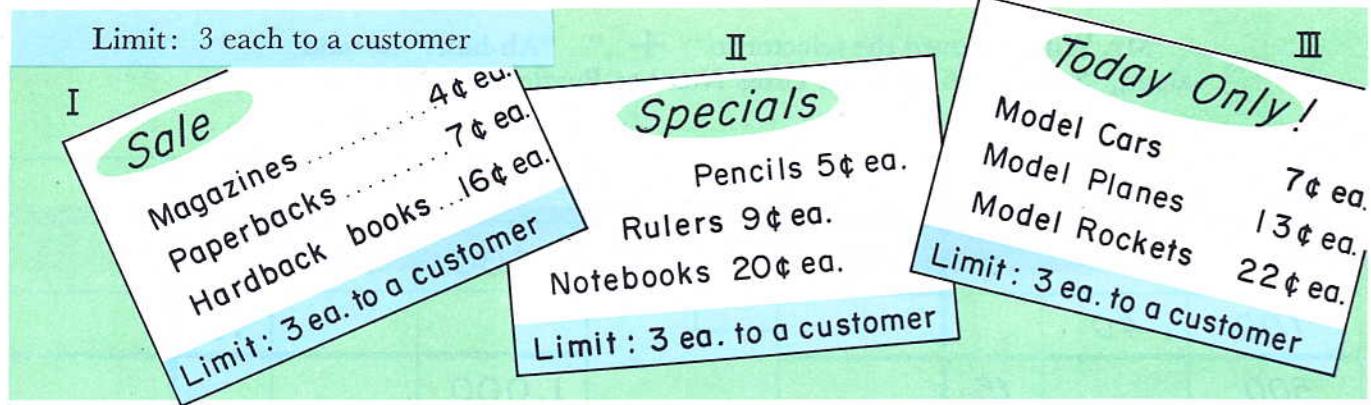
These signs refer to 3 different sales. We shall refer to them as Sale I, Sale II, and Sale III.

Could you spend exactly 36¢ under the terms of Sale II? The limit would rule out a purchase of 4 rulers at 9¢ each. A notebook costs 20¢. Since no combination of pencils and rulers costs 16¢, 36¢ could not be spent. The largest purchase without including a notebook is 3 pencils and 3 rulers for 42¢. One less pencil would make a purchase of _____ ¢. Three pencils and 2 rulers would cost _____ ¢.

It seems that exactly 36¢ can not be spent under the terms of Sale II.

Could exactly 36¢ be spent at Sale I? Yes: _____ magazines, _____ paperbacks, and _____ hardback books.

Could exactly 36¢ be spent at Sale III? Yes: _____ cars, _____ planes, _____ rockets.

> Spend exactly 39¢ at each sale.

Sale I:

_____ magazines, _____ paperbacks, _____ hardbacks

Sale II:

_____ pencils, _____ rulers, _____ notebooks

Sale III:

_____ cars, _____ planes, _____ rockets

Can exactly 39¢ be spent in any other way at any one of the sales? _____

> Spend exactly 53¢ at each sale.

Sale I:

_____ magazines, _____ paperbacks, _____ hardbacks

Sale II:

_____ pencils, _____ rulers, _____ notebooks

Sale III:

_____ cars, _____ planes, _____ rockets

> Spend exactly 62¢ at each sale.

Sale I:

_____ magazines, _____ paperbacks, _____ hardbacks

Sale II:

_____ pencils, _____ rulers, _____ notebooks

Sale III:

_____ cars, _____ planes, _____ rockets

In any one of the sales, can exactly 53¢ or 62¢ be spent in more than one way? _____

In any one of the sales, can an exact amount be spent in more than one way? _____

If you answered yes, list several examples.

_____ .... , _____ .... , _____ ....

_____ .... , _____ .... , _____ ....

> The largest purchase at each sale.

Sale I: _____ Sale II: _____ Sale III: _____

> The number of different purchases at a sale.

At Sale I, how many different purchases of 1 or more items could you make? _____ . Is the same true of Sales II and III? _____

A.

10	–	3	=	7
–		+		–
4	+	2	=	6
=		=		=
6	–	5	=	1

B.

15	–	8	=	7
+		+		+
5	–	2	=	3
=		=		=
20	–	10	=	10

C.

2	x	6	=	12
x		÷		x
9	÷	3	=	3
=		=		=
18	x	2	=	36

D.

8	x	6	=	48
÷		÷		÷
4	x	2	=	8
=		=		=
2	x	3	=	6

Watch the Signs!

Study the examples above.   WHAT ARE MY RULES?

1.

9	+	16	=	
+		–		+
12	–	7	=	
=		=		=
	+		=	

2.

24	+	16	=	
–		–		–
8	+	7	=	
=		=		=
	+		=	

3.

	+		=	55
–		+		–
27	–	19	=	
=		=		=
	+	31	=	

4.

5	x	6	=	
x		÷		x
12	÷	3	=	
=		=		=
	x		=	

5.

36	x	2	=	
÷		x		÷
9	÷	3	=	
=		=		=
	x		=	

6.

	x	4	=	
÷		÷		÷
	x		=	12
=		=		=
4	x		=	8

7.

52	–	18	=	
–		–		–
	–		=	16
=		=		=
27	–		=	

8.

30	+		=	38
+		+		+
	+	7	=	
=		=		=
	+		=	95

9.

	x		=	
x		x		x
	x	2	=	6
=		=		=
12	x		=	120

Please put in the signs and complete each example.

10.

31	---	18	=	
¦		¦		¦
24	---	15	=	
=		=		=
	---		=	4

11.

40	---		=	
¦		¦		¦
	---	7	=	
=		=		=
	---	18	=	42

12.

75	---	30	=	
¦		¦		¦
	---		=	5
=		=		=
91	---		=	

13

4 by 2 by 3

This is a 4 by 2 by 3 stack of blocks. You see exactly 3 faces of the D block, exactly 2 faces of the A, B, C, H, L, and P blocks. You see exactly one face of the E, F, G, I, J, K, M, N, O, T, and X blocks. There are 6 blocks you cannot see at all. If we follow the lettering scheme, these blocks would be the Q, R, S, U, V, and W blocks.

Here is a summary of this information:

	Exact number of faces showing				Total
	3	2	1	0	
4 by 2 by 3	1	6	11	6	24

Number of blocks

Make similar records for the stacks shown below and at the right.

	Exact number of faces showing				Total
	3	2	1	0	
4 by 2 by 3					
2 by 2 by 2					
3 by 3 by 2					
2 by 3 by 2					
3 by 5 by 2					
3 by 3 by 3					
3 by 4 by 8					
5 by 5 by 5					
4 by 4 by 4					
7 by 2 by 2					
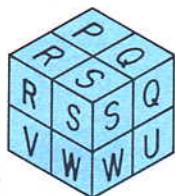
2 by 2 by 2
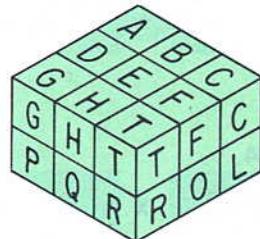
3 by 3 by 2
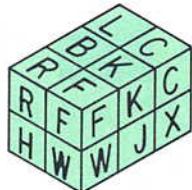
2 by 3 by 2
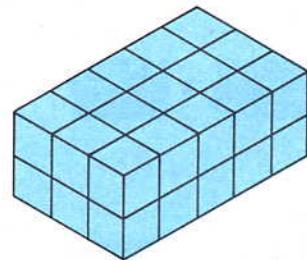
3 by 5 by 2

4 by 4 by 4

3 by 3 by 3
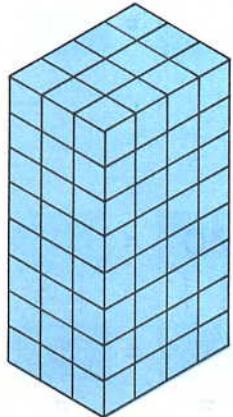
7 by 2 by 2

3 by 4 by 8
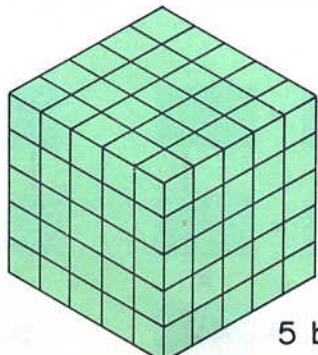
5 by 5 by 5
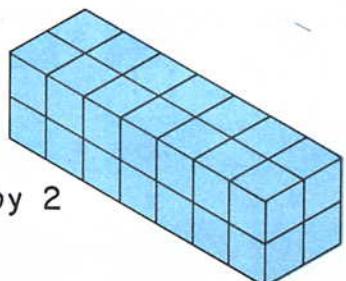
X	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0		
1	0	1	2	3	4	5	6	7	8	9			
2	0	2	4	6	8								
3	0	3	6	9									
4	0	4	8	12									
5	0	5	10										
6	0	6	12										
7	0	7											
8	0	8											
9	0	9											
10	0	10											
11	0	11											
12	0	12											
13	0	13											
14	0	14											
15	0	15											
16	0	16											
17	0	17											
18	0	18											
19	0	19											
20	0	20			80								
21	0	21											
22	0	22											
23	0	23											
24	0												
25													
26													
27													
28													
29													
30				90									

Mr. Dawson's Class Tackles a BIG JOB

"At first it was fun to learn how to multiply and divide," Marge began, "but now it's getting a little boring. I wish we had a computer."

Everyone in the class agreed.

"Well, why not make one," Mr. Dawson suggested.

"How?" everyone wanted to know.

"We can build a big multiplication table that will contain the product of any pair of 1- or 2-digit numbers from $0 \times 0 = 0$ to $99 \times 99 =$ _____."

"Great! When can we get started?"

With Mr. Dawson's help, the class tackled this big job.

Part of the table is shown on this page. Study the plan of the table. Please fill in the missing numbers.

Please use the table to complete the following sentences:

4 X 5 = ____   3 X 29 = ____

12 X 3 = ____   4 X ____ = 56

17 X 5 = ____   ____ X 5 = 90

4 X 16 = ____   15 X ____ = 60

5 X 23 = ____   29 X ____ = 145

27 X 4 = ____   ____ X 19 = 76

15

"How many entries will there be in our table?" Mr. Dawson asked.

"Well," Ben thought out loud, "there will be 100 entries in each row and there will be 100 rows."

$$100 \times 100 = \underline{\hspace{2cm}}$$

"That's too many multiplications," Marge protested.

Mr. Dawson said, "Let's work on one corner of the table and talk about our results."

"The diagonal of the table contains the familiar group of numbers called 'square numbers.' We'll talk about them later. But notice the entries on one side of the diagonal are just like those on the other. This is not surprising since we know that

$$3 \times 1 = 1 \times 3 \qquad 2 \times 3 = 3 \times 2$$
$$2 \times 5 = 5 \times 2 \qquad 4 \times 5 = 5 \times 4$$
$$\text{etc.}$$

"So, we can leave one side of the diagonal blank."

"Sure," Marge agreed. "On the last page, I looked up all the answers I needed in the part shaded in blue."

Jane had been working. She spoke up: "There are 10,000 less 100 that are not on the diagonal. Half of that number is $\underline{\hspace{1.5cm}}$."

Mr. Dawson had more words of encouragement. "Notice the column I've labeled 'a'— it's all zeros; and the column I've labeled 'b' is just a list of the whole numbers.

"I think I know why you point out the columns labeled 'c' and 'd'," Helen volunteered. "We learned how to count by 2's and 5's a long time ago. Including the one on the diagonal and those below it, there are $\underline{\hspace{1cm}}$ multiples of 2 and $\underline{\hspace{1cm}}$ multiples of 5 we can compute as fast as we can write."

Interest in working on the table increased as everyone began to see ways to make the job easier. Mr. Dawson had some words of advice before they started.

"Our first job will be to find shortcuts to avoid unnecessary computations."

"Our second job will be to look for quick ways to check our results. We don't want careless errors to get into our table."

"Finally, we will plan a way to share the work to avoid too much repetition."

16

Mark's Chain Reaction Idea

"I was experimenting with making a multiplication table.

"I notice that I could start with a couple of entries — and they would lead me to another. Here are two rows."

⟼ (beyond chart)

A.

X	2	3 →	5 →	8 →	13 →	→	→							
7	14	21 →	35 →	56										
9	18	27 →	45											

Top of columns: $2 + 3 = 5$; $3 + 5 = 8$; $5 + 8 = 13$; $8 + 13 =$ ____ , $13 +$ ___ $=$ ____

Row 7: $14 + 21 = 35$; $21 + 35 = 56$; $35 + 56 =$ ____ ; $56 +$ ____ $=$ ____

Row 9: $18 + 27 = 45$; $27 + 45 =$ ____ ; $45 +$ ___ $=$ ____ ; ____ $+$ ____ $=$ ____

"Then I noticed that I could add the entries in these two rows and get entries for row 16 because $7 + 9 = 16$."

⟼ (beyond chart)

B.

	2	3	5	8	13	21								
16	32	48												

$14 + 18 =$ ____ ; $21 + 27 =$ ____ ; $35 + 45 =$ ____ ; $56 +$ ____ $=$ ____

"Here are some more examples of my two-way Chain Reaction."

C.

X	7	10	17	27	44
12					
13					
25					
38					
63					

D.

X	20	3	23	26	49
9					
10					
19					
29					
48					

E.

X	7	20	27
5			
30			
35			

F.

X	11	22	33
6			
12			
18			

G.

X	8	16	24
9			
18			
27			

H.

X	12	10	22
12			
24			
36			

I.

X	22	10	32
36			
10			
46			

J.

X	32	2	34
46			
4			
50			

Paul's "Double or Add" System

"I use another kind of Chain Reaction that gives me every entry in a column," said Paul. "I double an entry or add a pair of neighbors. I'll use the column under 39 to show you my system.

"To get the entry in row 2, I DOUBLE the entry in row 1. To get the entry for row 3, I ADD THE NEIGHBORING entries in rows 1 and 2, etc."

I.

	39	
1	39	
2	78	.... double the entry in row 1
3	117	.... add the entries in rows 1 and 2
4	156	.... double the entry in row 2
5	195	.... add the entries in rows 2 and 3
6		.... double the entry in row 3
7		.... add the entries in rows 3 and 4
8		.... double the entry in row 4
9		.... add the entries in rows 4 and 5
10		.... double the entry in row 5

"At this point, I use what I already know to enter some checkpoints."

II.

1 x 39 = __39__   then   10 x 39 = __390__

2 x 39 = _____   then   20 x 39 = _____

3 x 39 = _____   then   30 x 39 = _____

4 x 39 = _____   then   40 x 39 = _____

5 x 39 = _____   then   50 x 39 = _____

etc.

Please enter the checkpoints and try using Paul's system.

III.

	39	
11	429	R5 and R6
12	468	double R6
13	507	R6 and R7
14		
15		
16		
17		
18		
19		Check-point ✓
20	780 ✓	
21		R10 and R11
22		
23		
24		
25		
26		
27		
28		
29		
30	1,170 ✓	
31		
32		
33		
34		
35		
36		
37		
38		
39		
40		✓

IV.

	39
41	
42	
43	
44	
45	
46	
47	
48	
49	
50	✓
82	
83	
84	
85	
86	
87	
88	
89	
90	✓
91	
92	
93	
94	
95	
96	
97	
98	
99	
100	✓

A.

	10	20	30	40	50	60	70	80	90	100
10	100									
20	200									
30										
40										
50										
60										
70										
80										
90										
100										

Checkpoints in the Table

Here are checkpoints which you will find at various locations in the table. Each number is the product of a pair of multiples of 10.

B.

	28	29	30	31	32	33		77	78	79	80	81	82
18			540										
19			570										
20	560		600										
21													
22													
57													
58													
59													
60													
61													
69													
70													
71													
72													

Certain columns and rows contain only multiples of 10.

"Mother made up a game," Harriet told the class. "She called it 'Fill up the holes in the table.'

"The game starts out easy. It becomes more difficult when the holes in the table are bigger and the numbers are larger. My system works so long as two entries are given."

Hint: Notice the differences between neighbors in the rows and columns.

a.

13
0
13
26
91

differences between neighbors: 13, 13, 13

b.

18	0	18	36	54						144	

differences between neighbors: 18, 18

c.

14	21		35

d.

		36	45	

e.

24
36
72

f.

15	18
	24
30	
	42

g.

24		40		56

h.

	50			80

i.

91			112	

j.

75	
	104
	182

k.

	144					207	

l.

51			102	
		90		

m.

36			
			63

n.

19		
	60	

o.

		44
	23	

p.

	38			95
		60		

q.

		494
476	504	
		589

r.

	172					430	

s.

58	
	120

t.

63	
	96

u.

3,431			3,572
			3,800

20

Ruth Studies the Diagonal

A.

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0												
1	1	1											
2		3	4										
3			5	9									
4													
5													
6													
7													
8													
9													
10													
11													
12													

"I found a chain reaction along the diagonal," said Ruth.

"I noted the differences between neighbors."

Ruth found these "differences between neighbors"

etc.

$$1 - 0 = \underline{\quad} \qquad 9 - 4 = \underline{\quad}$$
$$4 - 1 = \underline{\quad} \qquad 16 - 9 = \underline{\quad}$$
$$25 - 16 = \underline{\quad}$$
$$36 - 25 = \underline{\quad}$$

etc.

"My sister said that 1^2, 2^2, 3^2, 4^2, etc., is shorthand for 1×1, 2×2, 3×3, 4×4, etc. She said the shorthand is read 'one squared, two squared, three squared, four squared.'

"My tables look like this:"

B.

0 x 0	=	0^2 =	0
1 x 1	=	1^2 =	1
2 x 2	=	2^2 =	4
3 x 3	=	3^2 =	9
4 x 4	=	4^2 =	
5 x 5	=	5^2 =	
6 x 6	=	6^2 =	
7 x 7	=	7^2 =	
8 x 8	=	8^2 =	
9 x 9	=	9^2 =	

C.

10^2 =	✓
11^2 =	
12^2 =	
13^2 =	
14^2 =	
15^2 =	
16^2 =	
17^2 =	
18^2 =	
19^2 =	

D.

20^2 =	✓
21^2 =	
22^2 =	
23^2 =	
24^2 =	
25^2 =	
26^2 =	
27^2 =	
28^2 =	
29^2 =	

"I noticed a pattern for the units digits in each row of my table. I use this pattern to check my work."

$$6^2 = 3\underline{6} \qquad 16^2 = 25\underline{6} \qquad 26^2 = 67\underline{6}$$
$$3^2 = \underline{9} \qquad 13^2 = 16\underline{9} \qquad 23^2 = 52\underline{9} \text{ etc.}$$

Mr. Dawson's Hints

"You can easily compute the squares of multiples of 10 — 10^2, 20^2, 30^2, 40^2, etc.

"Here are some sketches that may help you find a shortcut for computing squares of the following multiples of 5 — 15^2, 25^2, 35^2, 45^2, etc."

A.

B.

"Can you find a rule to help you check the differences between neighbors?"

C.

6^2 =	36	
7^2 =	49	13

25^2 =		
26^2 =		

12^2 =	
13^2 =	

40^2 =	
41^2 =	

20^2 =	
21^2 =	

17^2 =	
18^2 =	

D.

30^2 = 0 ✓   61
31^2 = 1
32^2 = 4
33^2 = 9
34^2 = 6
35^2 = 5 ✓
36^2 = 6
37^2 = 9
38^2 = 4
39^2 = 1
40^2 = 0 ✓
41^2 =
42^2 =
43^2 =
44^2 =
45^2 = ✓
46^2 =
47^2 =
48^2 =
49^2 =
50^2 = ✓
51^2 =
52^2 =
53^2 =
54^2 =
55^2 = ✓
56^2 =
57^2 =
58^2 =
59^2 =
60^2 = ✓
61^2 =
62^2 =
63^2 =
64^2 =

E.

65^2 = ✓
66^2 =
67^2 =
68^2 =
69^2 =
70^2 = ✓
71^2 =
72^2 =
73^2 =
74^2 =
75^2 = ✓
76^2 =
77^2 =
78^2 =
79^2 =
80^2 = ✓
81^2 =
82^2 =
83^2 =
84^2 =
85^2 = ✓
86^2 =
87^2 =
88^2 =
89^2 =
90^2 = ✓
91^2 =
92^2 =
93^2 =
94^2 =
95^2 = ✓
96^2 =
97^2 =
98^2 =
99^2 =

The multiplication table grid with column headers 0–12 and row headers 0–12.

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0												
1		1											
2	0		4										
3		3		9									
4	0				16								
5			15										
6		3	12										
7													
8													
9													
10													
11													
12													

Step 6 Step 5 Step 4 Step 3 Step 2 Step 1

Another Chain Reaction

Fill in the blue blocks. Do you see a down and to the left pattern?

The first step down and to the left:

| 4 | 9 | 36 | 81 |

3

The entry at step 1 is _____ less than the entry on the diagonal.

The second step down and to the left:

| 4 | 16 | 25 | 49 |

0
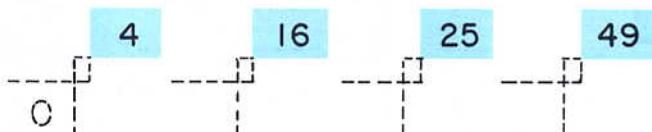
The entry at step 2 is _____ less than the entry on the diagonal.

The third step down and to the left:

| 16 | 64 | 81 | 121 |

7
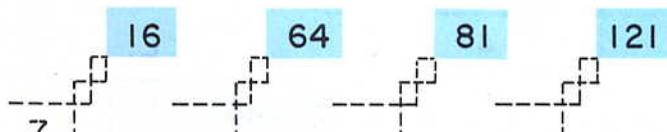
The entry at step 3 is _____ less than the entry on the diagonal.

The entry at step 4 is _____ less than the entry on the diagonal.

The entry at step 5 is _____ less than the entry on the diagonal.

The entry at step 6 is _____ less than the entry on the diagonal.

Here is another way to see this chain reaction.

8 x 8 = 64
9 x 7 = = 64 - 1
10 x 6 = = 64 - ____
11 x 5 = = 64 - ____
12 x 4 = = 64 - ____
13 x 3 = = 64 - ____

15 x 15 =
16 x 14 = 224 = -1
17 x 13 = = - ____
18 x 12 = = - ____
19 x 11 = = - ____

23
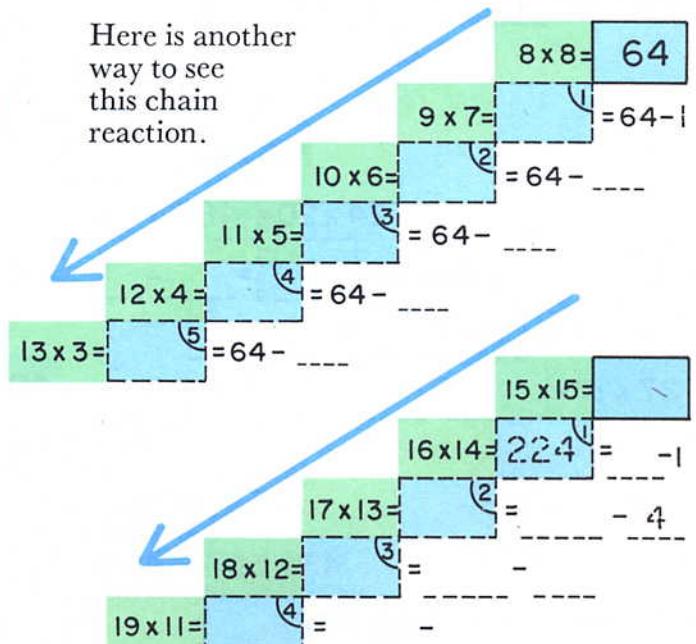
$7 \times 5 = (6+1) \times (6-1) = 36 - \underline{\quad}$

$10 \times 8 = (\quad +1) \times (\quad -1) = \underline{\quad} - \underline{\quad}$

$21 \times 19 = (\quad +1) \times (\quad -1) = \underline{\quad} - \underline{\quad}$

$24 \times 26 = (\quad -1) \times (\quad +1) = \underline{\quad} - \underline{\quad}$

$49 \times 51 = (\quad -1) \times (\quad +1) = \underline{\quad} - \underline{\quad}$

Second step down and left.

$10 \times 6 = (8+2) \times (8-2) = 64 - \underline{\quad}$

$13 \times 17 = (15-2) \times (15+2) = \underline{\quad} - \underline{\quad}$

$32 \times 28 = (\quad +2) \times (\quad -2) = \underline{\quad} - \underline{\quad}$

$47 \times 43 = (\quad +2) \times (\quad -2) = \underline{\quad} - \underline{\quad}$

Third step down and left.

$8 \times 2 = (5+3) \times (5-3) = 25 - \underline{\quad}$

$17 \times 23 = (20-3) \times (\quad +3) = \underline{\quad} - \underline{\quad}$

$15 \times 9 = (\quad +3) \times (\quad -3) = \underline{\quad} - \underline{\quad}$

Fourth step down and left.

$14 \times 6 = (10+4) \times (10-4) = \underline{100} - \underline{16}$

$7 \times 15 = (11-4) \times (11+\quad) = \underline{\quad} - \underline{\quad}$

$21 \times 29 = (\quad -4) \times (\quad +4) = \underline{\quad} - \underline{\quad}$

$11 \times 3 = (\quad +4) \times (\quad -4) = \underline{49} - \underline{\quad}$

Fifth step down and left.

$5 \times 15 = (10-5) \times (10+5) = \underline{100} - \underline{\quad}$

$17 \times 7 = (1 +5) \times (\quad -5) = \underline{\quad} - \underline{\quad}$

$25 \times 15 = (\quad -5) \times (\quad +5) = \underline{\quad} - \underline{\quad}$

Sixth step down and left.

$15 \times 3 = (9+6) \times (9-6) = \underline{81} - \underline{\quad}$

$14 \times 26 = (\quad -6) \times (\quad +6) = \underline{\quad} - \underline{\quad}$

$41 \times 29 = (\quad +6) \times (\quad -6) = \underline{\quad} - \underline{\quad}$

Use the difference of two squares as a short-cut in the following examples:

A.

$25 \times 25 = \boxed{}$

$26 \times 24 = [\quad] = \quad - \quad$

$27 \times 23 = [\quad] = \quad - 4$

$28 \times 22 = [\quad] = \quad - \underline{\quad}$

$29 \times 21 = [\quad] = \quad - \underline{\quad}$

$30 \times 20 = \boxed{600} = 625 - 25$

B.

$11 \times 9 = \underline{\quad} - \underline{\quad} = \underline{\quad}$

$29 \times 31 = \underline{\quad} - \underline{\quad} = \underline{\quad}$

$16 \times 14 = \underline{\quad} - \underline{\quad} = \underline{\quad}$

$11 \times 7 = \underline{81} - \underline{4} = \underline{\quad}$

$18 \times 22 = \underline{\quad} - \underline{\quad} = \underline{\quad}$

$12 \times 18 = \underline{\quad} - \underline{\quad} = \underline{\quad}$

$16 \times 8 = \underline{\quad} - \underline{\quad} = \underline{\quad}$

$35 \times 45 = \underline{\quad} - \underline{\quad} = \underline{\quad}$

$18 \times 6 = \underline{\quad} - \underline{\quad} = \underline{\quad}$

C.

$22 \times 22 = \boxed{}$

$23 \times 21 = [\quad] = \quad - \underline{\quad}$

$24 \times 20 = \boxed{480} = \quad - 4$

$25 \times 19 = [\quad] = \quad - \underline{\quad}$

$26 \times 18 = [\quad] = \quad - \underline{\quad}$

$27 \times 17 = [\quad] = \quad - \underline{\quad}$

D.

$39 \times 39 = \boxed{}$

$40 \times 38 = [\quad] = \quad - \underline{\quad}$

$41 \times 37 = [\quad] = \quad - \underline{\quad}$

$42 \times 36 = [\quad] = \quad - \underline{\quad}$

$43 \times 35 = [\quad] = \quad - \underline{\quad}$

$44 \times 34 = [\quad] = \quad - \underline{\quad}$

E.

$45 \times 45 = \boxed{}$

$46 \times 44 = [\quad] = \quad - \underline{\quad}$

$47 \times 43 = [\quad] = \quad - \underline{\quad}$

Completing CHUNKS of the TABLE . . . or, finding shortcuts

A.

X	5	6	7	9	10	11	15	16	17
6	30	36							
7	35	42							
8	40	48	56						
9									
10		60	70						
11	55								
16									
17									
18									

```
  6        10
 x 7      x 7

  16
 x 7

  6        10
x 10      x 10

  16
 x 10

  16
 x 17
  112
  160
  272
```

B.

X	7	8	9	19	20	21	27	28	29
4									
5									
6									
29									
30									
31									
34									
35									
36									

```
  8        20
 x 5      x 5

  28
 x 5

  8        20
x 30      x 30

  28
 x 30

  28
 x 35

   0
```

Use the table to save time in finding the missing numbers below:

```
  16        29         36         34         27          31
 x 7       x 30       x          x 27       x           x           x           x
 ____      ____       ____       ____       ____        ____        ____        ____
                      324                   918         899         256         841
```

25

X	7	8	9	29	30	31	32
5							
6							
7							
48							
49							
50							
51							

Fill in the missing digits. Use the completed table above if it will save you some time.

a.
```
    7
x 3 0
    0
```

b.
```
  2 1
x 4 0
```

c.
```
x 3 0
7 5 0
```

d.
```
    9
x
6 3 0
```

e.
```
  3
x   7
2 2
```

f.
```
x   7
6 3 0
```

g.
```
  9
x 7
```

h.
```
3 0
x 5
```

i.
```
x 6
7 2
```

j.
```
  7
x
5
```

k.
```
x 5
1 4 5
```

l.
```
  9
x
2
```

m.
```
x 7
9
```

n.
```
3 1
x
5 0
```

o.
```
  2
x 6
1 9
```

p.
```
3 3
x
1 9
```

q.
```
1 6
x
8
8 6 4
```

r.
```
x 2 8
4 6
```

s.
```
1 6
x
8 0
  4 8
```

t.
```
x
2 7 5
2 7 5
```

u.
```
x 6
5
```

v.
```
  8
x
3
```

w.
```
8
x
6
```

x.
```
x 9
3
```

y.
```
  1 7
x
    6
3 0
```

z.
```
    9
x 4
2
1 2 1
```

A.
```
  6 4
x
5 7
1 2 1
```

B.
```
x 5 1
1 6 0
      2
```

C.
```
2 9
x   9
1 4 2 1
```

D.
```
    8
x 3 2
1 5 3
```

E.
```
6 4
x
1 8 5 6
```

F.
```
x 4 7
3 9 9 5
```

G.
```
3 4
x
2 6 5 2
```

26

1.

X		
7	21	
8		32

3.

X		
	45	50
	54	

5.

X		
	40	48
9		54

2.

X		
4		36
5	40	

4.

X		
	0	7
		8

6.

X		
	56	64
		72

7.

X			
4		40	
10	70		
14			238

8.

X		30	
6			
9	18		
15			480

In exercises 9-11, the numbers to be written in the upper green box is the sum of the numbers in the white boxes. In exercises 12-17, each green box should contain the sum of the numbers in the white boxes to the left of it or above it.

9.

X	9	7	16
8			128

10.

X	2	11	
			91

11.

X	6		
	54		126

12.

X	8	5	13
9			
6			
15			

13.

X	7		
7			
5			216

14.

X	12		
6			
8			350

15.

X	20		
10			
	80		
			350

16.

X	10		
10			
12			228

17.

X		7	
		21	
9			
			228

Rule I: Same as bottom of the last page: the number to be written in a green box is the sum | of the numbers in the white boxes to the left of it or above it.

1.

X	9		15
5			75

2.

X	10		
7			91

3.

X		8	
4			152

Rule II: In any box that is checked (✓), write the largest multiple of ten you can and still be able to follow Rule I. The second and | third examples above follow Rule II, but the first does not.

4. ✓

X	10		
7			84

5. ✓

X			
5			290

6. ✓

X			
8			368

7. ✓

X			
9			612

8. ✓

X			
6			462

9. ✓

X			
12			144

10. ✓

X	0		
12			108

11. ✓

X			
15			375

12. ✓

X			
14			252

Rule II must be applied twice in each of the following:

13.

	20 ✓		
✓ 10			
13			299

14.

	✓		
✓			
26			910

15.

	✓		
✓ 40			760
		54	
			874

16.

	✓		
✓			
7	420		
27			1,701

17.

	✓		103
✓			
8	800		
			1,854

18.

	✓		
✓			2,850
		49	399
57			3,249

28

a.

X	
4	36

b.

X	
2	36

c.

X	6
	54

d.

X	5
	40

e.

X	
9	63

f.

X	12
10	

g.

X	
8	96

h.

X	
6	126

i.

	20
	100

j.

	40
5	

k.

	8
	64

l.

	11
9	

m. $7\overline{)56}$

n. $\overset{8}{}\overline{)72}$

o. $4\overline{)100}$

p. $2\overline{)130}$

q. $3\overline{)150}$

r. $10\overline{)130}$

s. $12\overline{)120}$

t. $25\overline{)300}$

u. $50\overline{)1{,}000}$

v. $30\overline{)600}$

w. $20\overline{)420}$

x. $20\overline{)500}$

X	9
5	45

$5\overline{)\overset{9}{45}}$ .... in shorthand

Use any correct method you know of to complete the following entries in the large chart. Look for shortcuts.

---

1. $8\overline{)72}$ ↕ **2.** $4\overline{)72}$

3. $25\overline{)800}$ ↕ **4.** $50\overline{)800}$

5. $70\overline{)910}$ ↕ **6.** $35\overline{)910}$

7. $40\overline{)}^{\,30}$ ↕ **8.** $80\overline{)1{,}200}$

9. $40\overline{)1{,}320}$ ↕ **10.** $20\overline{)1{,}320}$

11. $43\overline{)}^{\,21}$ ↕ **12.** $43\overline{)1{,}806}$

13. $26\overline{)78}$ → **14.** $26\overline{)520}$ → **15.** $26\overline{)}^{\,23}$ → **16.** $13\overline{)598}$ → **17.** $13\overline{)1{,}196}$ → **18.** $26\overline{)1{,}196}$

19. $72\overline{)360}$ → **20.** $72\overline{)2{,}160}$ → **21.** $72\overline{)2{,}520}$ → **22.** $36\overline{)}^{\,35}$ → **23.** $18\overline{)1{,}260}$ → **24.** $18\overline{)}^{\,140}$

25. $35\overline{)700}$ → **26.** $35\overline{)140}$ → **27.** $35\overline{)840}$ → **28.** $79\overline{)474}$ → **29.** $79\overline{)1{,}580}$ → **30.** $79\overline{)}^{\,26}$

31. $13\overline{)520}$ → **32.** $13\overline{)104}$ → **33.** $13\overline{)}^{\,48}$ → **34.** $26\overline{)624}$ → **35.** $39\overline{)624}$ → **36.** $52\overline{)624}$

I.

X					
3	21				351
6			102		
9				900	
27		270			

Skipping Around . . . in the table . . . and beyond the table.

LOOK FOR SHORTCUTS!

You may be able to find all the entries without even using scratch paper.

II.

X					
4					500
12				1,200	
24	120				
48		960	1,200		
72					

III.

X					
4			152		
16		480			
20					4,760
36	288				
18				3,600	

IV.

X				
8		80	72	
16				
24	7,200			
40				12,760

V.

X				
6				
70			560	
100		3,000		
176	17,600			24,288

The arrows are hints to shortcuts you may want to use.

a.

X	
14	420

b. 14) 98

c. 14) 518

d. 28) 5,180

e. 34) 3,400

f. 28) 420

g. 14) 196

h. 7) 196

i. 7) 56

j. 34) 1,700

k. 14) 840

l. 14) 392

m. 14) 1,232

n. 7) 784

o. 34) 272

p. 42) 840

q. 28) 784

r. 28) 56

s. 14) 1,568

t. 34) 158

What's My Rule?

Call out a number. I'll write it down. Then I'll use my rule and write a number under it.

When you discover my rule, you can fill in the blanks in each record.

a.

If you call out	7	15	0	11	29		
then I'll write	10	18	3			156

b.

	8	3	10	26	57			
	16	6	20				0	338

c.

	6	10	5	9	0		150
	19	31	16		46		

d.

	4	9	7	5			15
	16	81		64	36		

Now, I'll use an n, an a, or an x or whatever letter comes to my head to indicate the number you call out. I will include "pattern indicators" as clues to my rules.

1.

n	4	10	34	83	
$n + 17$	21	27			33

2.

a		10	4		27	118
$2a - 3$		17		27		

3.

x	3	11		8	40	
$x^2 + 1$	10		101			

4.

m		34	5	43		
$50 - m$		16	45		0	28

What Are My Rules?

Call out a number. I'll write it down. Then I'll use several rules and note the results. When you discover my rules you can use them to fill in the blanks in each record.

I.

x	6	10	8	15	0
$3x + 2x$	30				
$x^2 + 1$	37	101			
$5x$	30		40		

II.

a	6	10		16	
$a - 1$			20		49
$a \cdot (a-1)$					
$a^2 - a$					

III.

n	5	3	8		
$5 \cdot (n-1)$					
$3 \cdot (n+2)$					
$n^2 - n$				110	
$3n + 6$					
$n \cdot (n-1)$					
$5n - 5$					95
$6n \div 3$					

IV.

x		4			
$x + 3$					
$x - 3$			10		
$(x+3) \cdot (x-3)$					
x^2					100
$x^2 - 9$			40		
$x^2 + 6x + 9$					
$(x + 3)^2$					

Call out a pair of numbers. I'll use one or more rules and note some of the results. I'll use "pattern indicators" to show my rules. Please complete each record.

1.

a	7	10		23	
b	3	8	9		
a − b	4		0		47
a + b	10			41	85

2.

x	9	12	20		
y	3		9		
xy	27	60		25	63
x − y		7		0	2

3.

m	8	10	8		
n	5				
m + n	13	14		19	34
2m + n	21			31	
m + 2n	18		24		53

4.

f	5				
r	2		8	6	
$f^2 + r^2$	29	32		136	
$(f + r)^2$	49				625
$(f - r)^2$	9	0	1		9

5.

c	9	13			11
d	4		10		
c − d	5	7	8	0	
c + 2d	17			24	25
c · (d + 2)	54				
2c + 3d	30				
cd	36				
2cd − 1	71				

6. (make up your own)

a	7				
b	3				
2a − b	11				
2ab	42				
2(a − b)	8				
$(a - b)^2$	16				
$a^2 - b^2$	40				
$a^2 + b^2$	58				

Notice the green-tinted rows below. Perhaps you would like to try other numbers for *a* and *b* and for *s* and *t*. What are your conclusions?

7.

a	8	10	9		
b	3	4	7		
a − b	5				
$(a - b)^2$	25				
a^2	64				
b^2	9				
2ab	48				
$a^2 - 2ab + b^2$	25				

8.

s		5	13		
t		4	2		
s + t					
$(s + t)^2$					
s^2					
t^2					
2st					
$s^2 + 2st + t^2$					

I.
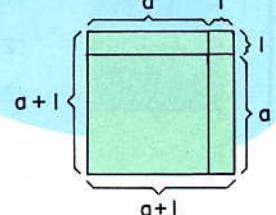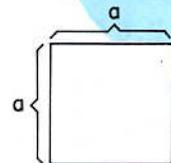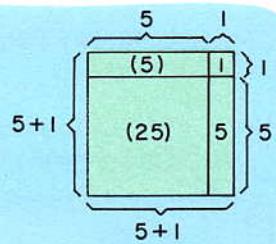
a	5	10					
a + 1	6			15			41
$(a+1)^2$	36		81				
a^2	25					625	
2a + 1	11				39		
$a^2 + 2a + 1$	36						

Glance back at pages 20 and 21. Moving from a^2 to $(a + 1)^2$ is "moving one step down the diagonal."

II.
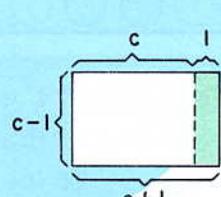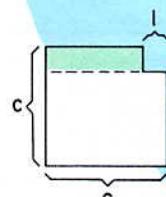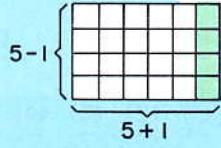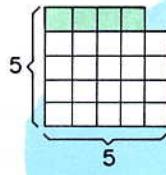
a	5						
a^2	25	64					
$a^2 - 1$	24		99				
a + 1	6		16				
a − 1	4			24			
$(a+1)\cdot(a-1)$	24				675		

Glance back at pages 22 and 23 . . . 1 step down and left.

III.
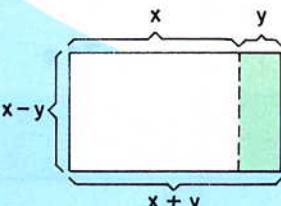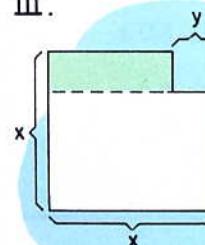
r	6	8					
s	4	3					
r^2	36	144					
s^2	16	25	81				
$r^2 - s^2$	20		319				
r + s	10			95			
r − s	2			35			
$(r+s)\cdot(r-s)$	20						

Glance again at pages 22 and 23 . . . s steps down and left.

33

27
x 7

	(1)	(2)	(3)	(4)
	199	189	34	207

Mark the correct answer

Mark the furthest from correct answer

(1) (2) (3) (4)
○ ● ⊗ ○

155 ── Difference between the answers you marked

1

36
x 24

(1)   216
(2)   774
(3) 1,444
(4)   864

(1) (2) (3) (4)
○ ○ ○ ○ ─────

2

$37.28
−18.94

(1) $ 56.22
(2) $ 18.34
(3) $ 28.34
(4) $ 18.94

(1) (2) (3) (4)
○ ○ ○ ○ $ ─────

3

1,867
  170
  568
+ 77

(1) 2,826
(2) 2,862
(3) 2,628
(4) 2,682

(1) (2) (3) (4)
○ ○ ○ ○ ─────

4

13) 221

(1)   17
(2) 108
(3)   38
(4)   27

(1) (2) (3) (4)
○ ○ ○ ○ ─────

5

$(3 \times \square) - \square = 36$

(1) $\boxed{12}$ (3) $\boxed{108}$
(2) $\boxed{18}$ (4) $\boxed{39}$

(1) (2) (3) (4)
○ ○ ○ ○ ─────

6

496
x 8

(1) 3,228
(2) 4,968
(3) 3,968
(4) 3,758

(1) (2) (3) (4)
○ ○ ○ ○ ──────

Prices at the Supermarket

Grapefruits : 10¢ each , 3 for 28¢
Bananas : 14¢ a lb., 2 lbs. for 25¢
Oranges : 48¢ a dozen

Potatoes : 7¢ a lb.
        10 lbs. 59¢
        25 lbs. $1.25

7 8

Mr. Wilson bought a dozen grapefruit. He paid:

(1) $ 1.12   (3) 84¢
(2) $ 1.20   (4) $ 2.80

(1) (2) (3) (4)
○ ○ ○ ○ $ ─────

He also bought 30 pounds of potatoes. They cost him:

(1) $ 2.10   (3) $ 1.60
(2) $ 1.77   (4) $ 1.98

(1) (2) (3) (4)
○ ○ ○ ○ ───── ¢

9

Mrs. Rand bought half a dozen oranges and 4 pounds of bananas. She paid:

(1) 88 ¢   (3) 50 ¢
(2) 74 ¢   (4) 24 ¢

(1) (2) (3) (4)
○ ○ ○ ○ ───── ¢

10 11

What would be the cost of 5 grapefruit and 7 pounds of bananas?

(1) $ 1.48   (3) $ 1.55
(2) $ 1.20   (4) $ 1.37

(1) (2) (3) (4)
○ ○ ○ ○ ───── ¢

What would be the cost of $2\frac{1}{2}$ dozen oranges, 20 pounds of potatoes, and half a dozen grapefruit?

(1) $ 3.17   (3) $ 2.90
(2) $ 2.94   (4) $ 2.49

(1) (2) (3) (4)
○ ○ ○ ○ ───── ¢

12

What would be the total cost of 15 pounds of potatoes and 15 pounds of bananas?

(1) $ 2.83   (3) $ 4.50
(2) $ 3.25   (4) $ 5.02

(1) (2) (3) (4)
○ ○ ○ ○ $ ─────

I

three hours .... ▨

one and one-quarter hours .... ▨

fifty minutes .... ▨

difference in minutes between

A and B: _ _ _ _ _ _ min.   B and C: _ _ _ _ _ _ min.

A and C: _ _ _ _ _ _ min.

II

two dollars and sixty-seven cents .... ▨

three-quarters of a dollar ....

four and one-half dollars .... ▨

difference in cents between

A and B: _ _ _ _ _ _ ¢   B and C: _ _ _ _ _ _ ¢

A and C: _ _ _ _ _ _ ¢

III

two yards .... ▨

two and one-half feet .... ▨

one yard, one foot, and seven inches .... ▨

one-half yard .... ▨

difference in inches between

A and B: _ _ _ _ _ _ in.   B and D: _ _ _ _ _ _ in.

B and C: _ _ _ _ _ _ in.   C and D: _ _ _ _ _ _ in.

A and C: _ _ _ _ _ _ in.   A and D: _ _ _ _ _ _ in.

IV

thirty-one ounces .... ▨

three-quarters of a pound .... ▨

four pounds .... ▨

one and one-half pounds .... ▨

difference in ounces between

A and B: _ _ _ _ _ _ oz.   B and D: _ _ _ _ _ _ oz.

B and C: _ _ _ _ _ _ oz.   C and D: _ _ _ _ _ _ oz.

A and C: _ _ _ _ _ _ oz.   A and D: _ _ _ _ _ _ oz.

V

five thousand, seven hundred .... ▨

five thousand, seven .... ▨

five thousand, nineteen .... ▨

five thousand, ninety .... ▨

five thousand, one hundred ninety .... ▨

five thousand, seventy .... ▨

difference between

A and F: _ _ _ _ _ _   B and D: _ _ _ _ _ _ _

E and C: _ _ _ _ _ _ _   A and C: _ _ _ _ _ _ _

C and D: _ _ _ _ _ _ _   F and C: _ _ _ _ _ _

A and D: _ _ _ _ _ _ _   F and B: _ _ _ _ _ _

B and E: _ _ _ _ _ _   A and E: _ _ _ _ _ _

VI

eight thousand, sixty-seven .... ▨

forty thousand, three hundred .... ▨

nine thousand, eight hundred fifty .... ▨

ten thousand, nine hundred eight .... ▨

fifteen thousand .... ▨

six thousand, nine .... ▨

difference between

D and F: _ _ _ _ _ _   A and B: _ _ _ _ _ _

C and D: _ _ _ _ _ _   C and F: _ _ _ _ _ _

B and C: _ _ _ _ _ _   B and D: _ _ _ _ _ _

E and F: _ _ _ _ _ _   C and E: _ _ _ _ _ _

A and D: _ _ _ _ _ _ _   A and C: _ _ _ _ _ _

Select one number from each list to find the
right combination to complete the table.

I

	List										
A	0 , 1 , 3 , 4 , 6 , 7 , 9	6	4	0	7						
B	0 , 10 , 21 , 30 , 43 , 57	21	43	10	0	43					
	Totals →	27	47	10	7	52	25	60	36	22	66

II

	List								
C	0 , 1 , 2 , 5 , 7 , 8	2							
D	0 , 9 , 18 , 28 , 37 , 47	18	9						
E	0 , 56 , 113 , 169 , 225 , 284	56	0						
	Totals	76	16	174	131	280	322	92	213

III

	List								
F	0 , 2 , 4 , 6 , 8								
G	0 , 9 , 19 , 27 , 36	36							
H	0 , 45 , 93 , 138 , 184	93	184						
I	0 , 237 , 466 , 695 , 924	237	695	924					
	Totals	370	885	980	815	538	481	406	163

IV

	List							
J	0 , 1 , 2 , 4							
K	0 , 5 , 11 , 16							
L	0 , 21 , 43 , 67							
M	0 , 88 , 179 , 267	0						
N	0 , 358 , 705 , 1059	358						
	Totals	440	820	441	1,344	848	650	1,277

Tom announced: "I've got a way to find
these combinations quickly. I simply take the
largest one I can from the bottom list; next,
the largest one from the next list, etc. I keep
a running account of how many I have left to
go:

Total [1,2 7 7]
1 0 5 9   Largest from List N
2 1 8 . . . amount needed
1 7 9   Largest from List M
3 9 . . . amount still needed
2 1   Largest from List L
1 8 . . . amount still needed
1 6   Largest from List K
2 . . . amount still needed
2   From List J

Use lists J, K, L, M, and N to try Tom's
system on the following:

	V	**VI**	**VII**
Totals	957	672	1,206
N	705		
	252		
M	179		
	73		
L			
K			
J			

Mr. Stone operated a chicken ranch. His main business was supplying eggs to the stores in his vicinity.

He packed eggs by the dozen and by the gross. (A gross is 12 dozen.) If a customer ordered 15 dozen eggs, he would ship 1 gross and 3 dozen.

Here are examples of special order cards that were filled when orders came in:

A				
Eggs	●	●	●	●
	●	●		
Doz.				
Gross	●			
Total: 150				

B				
Eggs	●	●	●	●
	●	●	●	
Doz.	●	●		
Gross	●			
Total: 175				

C				
Eggs				
Doz.	●			
Gross	●	●		
Total: _____				

Please fill in the total number of eggs on order card C above.

The shipping department would use these cards to determine how many gross, how many dozen, and how many additional eggs (an incomplete dozen) to send to the customer whose name was on the back of the card.

Please show how cards D, E, and F should be marked.

D				
Eggs				
Doz.				
Gross				
Total: 100				

E				
Eggs				
Doz.				
Gross				
Total: 516				

F				
Eggs	●	●	●	●
Doz.	●	●	●	●
	●	●		
Gross	●	●	●	●
Total: _____				

As a time saver, someone in the office made a chart similar to the following. Please supply the missing entries.

Incomplete Dozen	Dozen		Gross	
1	1	12	1	144
2	2	24	2	288
3	3	36	3	
4	4		4	
5	5		5	
6	6		6	
7	7		7	
8	8		8	
9	9		9	
10	10		10	
11	11		11	

An assistant to Mr. Stone designed a card that was easier to use. He told Mr. Stone, "Everyone knows where we mark the incomplete dozens, the dozens, the gross, and total. We can leave the words out. Also, we can circle a number faster than we can put in dots."

Please complete the new order cards G, H, I, and J. Card G is already completed to illustrate the simplified method. (The chart above may help.)

G				H				I				J			
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
5	6	7	8	5	6	7	8	5	6	7	8	5	6	7	8
⑨	10	11	12	9	10	11	12	9	10	11	12	9	10	11	12
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
5	⑥	7	8	5	6	7	8	5	6	7	8	5	6	7	8
9	10	11	12	9	10	11	12	9	10	11	12	9	10	11	12
①	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
5	6	7	8	5	6	7	8	5	6	7	8	5	6	7	8
9	10	11	12	9	10	11	12	9	10	11	12	9	10	11	12
225				1,500				550				1,000			

G: 1 gross, 6 dozen, and 9 eggs; or 225 eggs

37

Please complete the following orders for eggs:

K

1	2	3	4
5	6	7	8
9	10	11	12
1	2	3	4
5	6	7	8
9	10	11	12
1	2	3	4
5	6	7	8
9	10	11	12

140

L

1	2	3	4
5	6	7	8
9	10	11	12
1	2	3	4
5	6	7	8
9	10	11	12
1	2	3	4
5	6	7	8
9	10	11	12

300

M

96

N

275

O

375

P

475

Q

575

One of Mr. Stone's customers was Mrs. Carter's Lunch Room.

Mrs. Carter liked the new egg order cards so much that she decided to make checks that had amounts printed on them. The clerks learned how to circle the right amounts:

$ 0.00	$ .00	$ .00
1.00	.10	.01
2.00	.20	.02
(3.00)	.30	.03
4.00	.40	(.04)
5.00	.50	.05
6.00	.60	.06
7.00	(.70)	.07
8.00	.80	.08
9.00	.90	.09

This is marked to show $3.74

Mrs. Carter simplified the forms:

A

$0	0	0
1	1	1
2	2	2
(3)	3	3
4	4	(4)
5	5	5
6	6	6
7	(7)	7
8	8	8
9	9	9

$ 3.74

B

$0	(0)	0
1	1	1
2	2	2
3	3	3
(4)	4	4
5	5	5
6	6	6
7	7	7
8	8	(8)
9	9	9

$_____

C

($0)	0	(0)
1	1	1
2	2	2
3	3	3
4	4	4
5	5	5
6	6	6
7	7	7
8	8	8
9	(9)	9

$_____

At the end of the day, she often collected the checks that were for the same amount of money and put them in a pile. For each pile, she made a note of

(1) the number of checks that were in the pile, and

(2) the total for the pile.

Please fill in the missing information below.

D

$0	0	0
1	(1)	1
(2)	2	2
3	3	3
4	4	4
5	5	(5)
6	6	6
7	7	7
8	8	8
9	9	9

7 checks

$_____ Total

E

$0	0	0
1	1	1
2	2	2
3	3	3
4	4	4
5	5	5
6	6	6
7	7	7
8	8	8
9	9	9

13 checks

$ 16.25 Total

F

$0	0	0
1	1	1
2	2	2
3	3	3
4	4	4
5	5	5
6	6	6
7	7	7
8	8	8
9	9	9

16 checks

$ 26.72 Total

Please circle the correct amounts on the top check in the E and F piles above in the way all checks in each pile must be marked.

Tom was excited: "I can use my system again."

(continued)

"Examine the piles of checks I-VI. Then I'll tell you how I find the missing information."

	I			II			III			IV			V			VI	
$0	0	0	$0	0	0	$0	0	0	$0	0	0	$0	0	0	$0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9

- I: 17 checks — $ 39.95 Total
- II: 14 checks — $ 25.76 Total
- III: 11 checks — $ 40.92 Total
- IV: 18 checks — $ 45.00 Total
- V: 31 checks — $ 29.45 Total
- VI: 7 checks — $ 40.04 Total

"I need to account for $39.95, the total of the checks in pile I. There are 17 checks in this pile and they are all alike.

"First, I wonder which number should be circled in the green column of each check. If 1 is circled, that would account for $17 of the total; if 2 is circled, that would account for $34; and if 3 is circled, that would account for $51 — which is too much.

"So, 2 must be circled. This accounts for $34 of the $39.95. I keep notes like this:

$ 39.95 ...... Total
34.00 ...... from green column
5.95 ...... to be accounted for

"Next, I consider the blue (dimes) column:

$ .10 x 17 = $1.70
.20 x 17 = _____
.30 x 17 = _____
.40 x 17 = _____ (too much!)

"Now I know that 3 must be circled in the blue column. I note the following:

$ 5.95 ...... (to be accounted for)
_____ ...... from blue column
_____ ...... still to be accounted for

"So, ☐ must be circled in the white column, since $_____ × 17 = _____.

"Here are my notes on piles II and III." Please fill in the missing entries.

$ 25.76 .... total of 14 checks
_____ ...... $1.00 × 14
...... to be accounted for
_____ ...... $0.80 × 14
...... to be accounted for
_____ ...... $0.04 × 14

"Pile number III:

$ 40.92 .... total of 11 checks
_____ ...... $3.00 × 11
_____ ...... $0.70 × 11
_____ ...... $0.02 × 11

Please find the missing entries for piles IV through VI.

39

Martin's System

"I've got some lists that I can use any time I run into certain problems. If I need to divide by 23, I take out my 'twenty-three' lists. Here they are."

		0	100	200	300	400	500	600	700	800	900
A	23	0	2,300								

		0	10	20	30	40	50	60	70	80	90
B	23	0	230	460							

		0	1	2	3	4	5	6	7	8	9
C	23	0	23	46	69						

"Here is how I do the following problems."

(1)
$$23 \overline{)5,681} = 247$$

(2)
$$23 \overline{)10,948}$$

(3)
$$23 \overline{)17,457}$$

(4)
$$23 \overline{)18,584}$$

"I select three numbers — one from List A, one from List B, one from List C — so that their sum is the dividend. I notice the number above each selection and write it in the right-hand column."

(1)

A →	4,600	200
B →	920	40
C →	161	7
	5,681	247

(2)

A →		
B →		
C →		
	10,948	

(3)

A →		
B →		
C →		
	17,457	

(4)

A →		
B →		
C →		
	18,584	

Please use Martin's system to complete each of the 4 examples below:

(5)
$$23 \overline{)3,749}$$

(6)
$$23 \overline{)9,131}$$

(7)
$$23 \overline{)13,340}$$

(8)
$$23 \overline{)2,277}$$

(5)

A →		
B →		
C →		
	3,749	

(6)

A →		
B →		
C →		
	9,131	

(7)

A →		
B →		
C →		
	13,340	

(8)

A →		
B →		
C →		
	2,277	

40

Martin Develops a Shorthand

"I found a shorter way to use my system. I keep track as I go along.

"Consider the first example in the previous group. I look for the largest number in any list that is less than the dividend. I write it down this way — and find out how much I have left to take care of.

(1)

```
        2 0 0  ←----------┐
   23 | 5,6 8 1           |
        4 6 0 0  ← largest number in list A
        1 0 8 1  ← left to take care of
```

"Next, I find the largest number I can use in list B, and extend my record.

```
          4 0  ←--------┐
        2 0 0           |
   23 | 5,6 8 1         |
        4 6 0 0         |
        1 0 8 1         |
          9 2 0  ← largest number in list B
            6 1  ← left to take care of
```

"Then

```
            7  ←------┐
          4 0         |
        2 0 0         |
   23 | 5,6 8 1       |
        4 6 0 0       |
        1 0 8 1       |
          9 2 0       |
            6 1       |
            6 1  ← largest number is list C
```

"Here is the way I would find the quotient in some of the other examples."

(2)

```
        4 0 0  ←----┐
   23 | 0,9 4 8   ← A
        9 2 0 0   ←┤
        1 7 4 8    ├ B
                   |
                   └ C
```

(3)

```
   23 | 7,4 5 7
        1 6 1 0 0
          3 5 7
```

(4)

```
              0  ←----┐
   23 | 8,5 8 4       |
                      ├ B
              0  ←----┘
          1 8 4
```

(5)

```
   23 | 3,7 4 9
```

(6)

```
   23 | 9,1 3 1
```

(7)

```
                0
   23 | 3,3 4 0
                      0
```

Alice objected: "But Martin, your system only works when you can use your 23-lists!"

"No, Alice, I have other lists — an 18-list, for example."

"Here are my 18-lists," Martin explained.

		0	100	200	300	400	500	600	700	800	900
A	18	0	1,800	3,600							

		0	10	20	30	40	50	60	70	80	90
B	18	0	180								

		0	1	2	3	4	5	6	7	8	9
C	18	0	18	36							

"Notice that I begin to leave out zeros in those places where I know zeros would normally be written!"

(1)
```
          2 0
   18 | 4 1 4   B
        3 6 0
          5 4   C
          5 4
```

(2)
```
   18 | 1,1 1 6
```

(3)
```
   18 | 3 0 6
        1 8 0
            6
```

(4)
```
   18 | 1,6 9 2
              0
              2
```

(5)
```
   18 | 2,4 4 8
              0 0
          6 4
              0
              8
```

(6)
```
            0
          0 0
   18 | 8,6 0 4
            0 0
        1 4 0 4
        1 2 6 0
        1 4 4
```

(7)
```
   18 | 4,5 1 8
            1
            8
```

(8)
```
   18 | 1 2,3 6 6
          1 5 6
```

(9)
```
   18 | 1 6,9 5 6
              5
```

(10)
```
   18 | 9,6 4 8
```

(11)
```
   18 | 1 5,4 9 8
```

(12)
```
   18 | 6,3 7 2
```
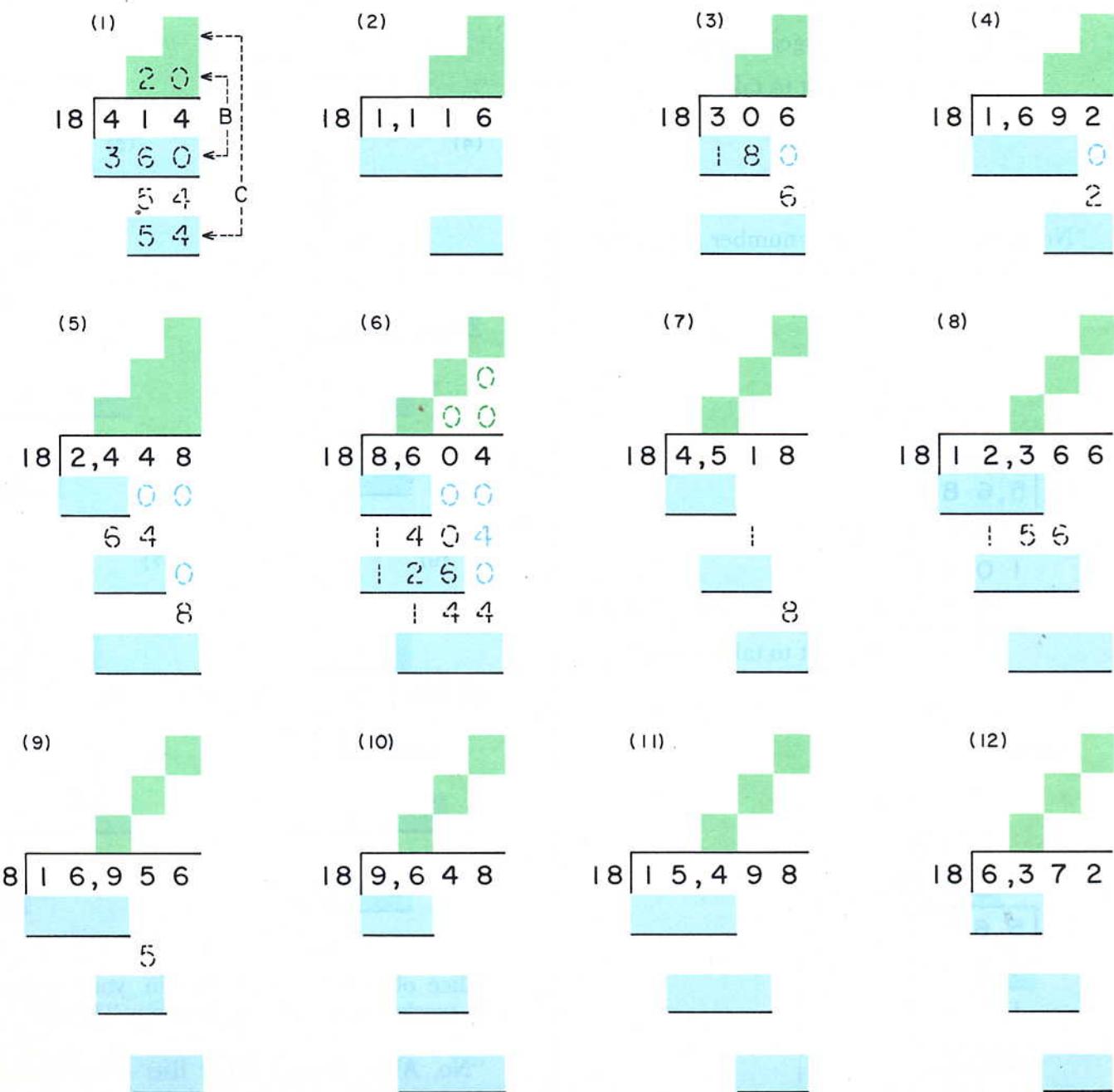
"My slogan is: 'When in doubt, don't leave the zeros out!'"

"At this point," Martin explained, "when I stopped writing so many zeros, I realized that list C was the only list I needed for my work.

"Here are some examples using just one 37-list."

37	0	1	2	3	4	5	6	7	8	9
	0	37								

a.
```
     2 0
37 | 8 5 1
       0
       1
```

b.
```
37 | 2,9 9 7
          7
```

c.
```
37 | 1,9 2 4
```

d.
```
37 | 4 0 7
```

e.
```
37 | 4,7 3 6
         0
```

f.
```
37 | 2 8,1 9 4
           0
```

g.
```
37 | 1 9,6 1 0
```

h.
```
37 | 1 5,0 9 6
             9
          0 0
             6
```

"I improved the shorthand as shown in these examples."

i.
```
37 | 2,7 3 8
     2 5 9
         8
```

j.
```
         3
37 | 1 3,5 0 5
```

k.
```
37 | 3 4,8 5 4
```

l.
```
37 | 6,6 9 7
```

m.
```
37 | 9,4 7 2
```

n.
```
37 | 1 7,7 9 7
```

o.
```
37 | 2,5 5 3
```

p.
```
37 | 1 1,3 5 9
             5
          0 0
```

43

"For the examples below," Martin explained, "I prepared several lists and used the one I needed for each example."

Below are three of Martin's lists.

	0	1	2	3	4	5	6	7	8	9
15	0	15	30							135
49	0							343		
62	0			186	248				496	

I. 15 | 7 6 5

II. 49 | 1,0 2 9

III. 62 | 5,3 9 4

IV. 49 | 2,8 4 2

"Notice my shortcut when I use zero from one of my lists."

V. 62 | 1 6,1 2 0

VI. 49 | 1 5,0 9 2

VII. 15 | 5,5 5 0

VIII. 49 | 2 4,9 4 1

IX. 49 | 3 4,4 9 6

X. 62 | 1 7,3 6 0

XI. 15 | 1 2,0 9 0

XII. 49 | 4 4,2 4 7

VIII. 62 | 2 3,4 3 6

XIV. 15 | 1 4,4 7 5

XV. 49 | 2 9,6 4 5

XVI. 62 | 4 2,9 0 4

44

"Suppose you forget or lose your lists? What do you do then?" Carol asked.

"Oh, I just multiply as I go along," Martin began. "I'll show you what my work looks like."

(Please fill in the missing digits.)

1.
```
            4
          6 0
        3 0 0
   12 ) 4,3 6 8
        3 6 0 0
          7 6 8
          7 2 0
            4 8
            4 8
```

2.
```
   19 ) 2,1 4 7
          0 0
              7
            0
```

3.
```
              0
            0 0
   13 ) 6,1 3 6
        5 2 _
          9 3 _
          9 1 _
            2 6
```

4.
```
   16 ) 6,9 6 0
```

5.
```
   31 ) 1 7,4 2 2
```

6.
```
   24 ) 1 7,8 8 0
```

7.
```
   38 ) 1 5,8 8 4
```

8.
```
   25 ) 9,4 5 0
```

9.
```
   91 ) 4,7 3 2
```

10.
```
   27 ) 1,9 7 1
```

11.
```
   43 ) 3,4 4 0
```

12.
```
   19 ) 5 1 3
```

13.
```
   52 ) 2 0,0 2 0
```

14.
```
   18 ) 1 1,4 1 2
```

15.
```
   34 ) 2 5,3 6 4
```

16.
```
   21 ) 1 6,9 4 7
```

"Here are some lists I made for solving problems involving larger numbers."

P. X	127
0	0
1	
2	
3	
4	
5	
6	
7	
8	1,016
9	1,143

Q. X	127
0	0
10	
20	
30	
40	
50	
60	
70	8,890
80	10,160
90	11,430

R. X	127
0	0
100	
200	
300	
400	50,800
500	63,500
600	76,200
700	88,900
800	101,600
900	114,300

S. X	127
0	0
1000	
2000	
3000	381,000
4000	508,000
5000	635,000
6000	762,000
7000	889,000
8000	1,016,000
9000	1,143,000

T. X	127
0	0
10,000	
20,000	
30,000	3,810,000
40,000	5,080,000
50,000	6,350,000
60,000	7,620,000
70,000	8,890,000
80,000	10,160,000
90,000	11,430,000

"Suppose that I wanted to solve the following problems."

1.

$$127\overline{)1,5\ 7\ 7,4\ 6\ 7}$$

2.

$$127\overline{)1,0\ 1\ 1,8\ 0\ 9}$$

3.

$$127\overline{)}\quad 4\ 3,2\ 8\ 5$$

"I would select several numbers — one from list P, one from list Q, one from list R, and so on — until their sum is the dividend."

(Please find the missing entries.)

T.	1,270,000	10,000
S.		2,000
R.	50,800	
Q.		
P.		1
	1,577,467	

	0	0
	889,000	
	114,300	
	7,620	
	1,011,809	

		43,285

Here are the same problems — with only the first steps — started in different shorthands. Please supply the missing entries.

4.

$$127\overline{)1,5\ 7\ 7,4\ 6\ 7}$$

5.

$$127\overline{)1,0\ 1\ 1,8\ 0\ 9}$$

6.

$$127\overline{)}\quad 4\ 3,2\ 8\ 5$$

In the following examples, just record the entries
for the first step — as the first example indicates.

a.
$$\begin{array}{r} 3 \\ 127\overline{)453{,}644} \\ 3\ 8\ 1 \end{array}$$

b.
$$127\overline{)8{,}143{,}113}$$

c.
$$127\overline{)1{,}253{,}490}$$

d.
$$127\overline{)2{,}413}$$

e.
$$127\overline{)124{,}173}$$

f.
$$127\overline{)9{,}867{,}910{,}160}$$

In the following examples, record the entries for the first two steps.

g.
$$127\overline{)66{,}929}\ ^{5}$$

h.
$$127\overline{)253{,}873}$$

i.
$$127\overline{)125{,}603{,}254}$$

j.
$$127\overline{)90{,}043}$$

k.
$$127\overline{)8{,}382}$$

l.
$$127\overline{)9{,}906}$$

m.
$$127\overline{)1{,}143}$$

Complete the following examples:

n.
$$127\overline{)1{,}577{,}467}$$

o.
$$127\overline{)1{,}011{,}809}$$

p.
$$127\overline{)446{,}786}$$

When in doubt, don't leave the zeros out!

Here are several of Martin's lists. Please fill in the missing entries.

X	0	1	2	3	4	5	6	7	8	9
19	0	19	38							
43	0									
65	0									
209	0									
758	0	758	1,516							

If you wish, you can use the tables to help complete
some of the examples on this page and the next page.

(1)

43 | 2 1 5,2 5 8

(2)

65 | 1 4 9,5 0 0

(3)

19 | 1 8 6,5 2 3

(4)

38 | 1 5 2

(5)

209 | 6 2,9 0 9

(6)

43 | 1 7,5 0 1

(7)

19 | 1 8,0 8 8

(8)

758 | 3 1,0 7 8

(9)

209 | 1,9 1 8,8 2 9

(10)

22 | 6 8 2

(11)

65 | 1 3,3 9 0

(12)

43 | 3 5 8,1 4 7

(13)

758 | 1 3 1,8 9 2

48

(14)
15 | 1,140

(15)
29 | 986

(16)
81 | 9,639

(17)
43 | 116,315

(18)
19 | 6,916

(19)
55 | 2,475

(20)
16 | 8,448

(21)
209 | 72,105

(22)
24 | 17,136

(23)
78 | 5,534,100

(24)
25 | 211,550

(25)
37 | 40,330

(26)
85 | 59,500

(29)
758 | 2,665,886

(27)
65 | 63,115

(28)
120 | 114,960

(30)
2 | 1,561,008

(31)
8 | 100,000

(32)
7 | 10,000,004

49

Surprises

"Here's a trick," Mary explained to Bill, "that has an interesting surprise to it. Mark three points on your piece of paper and draw a line through each pair of points. Then cut along those lines and a piece of the paper will drop out."

"Sure," said Bill, "and the pieces that drop out will be triangular regions."
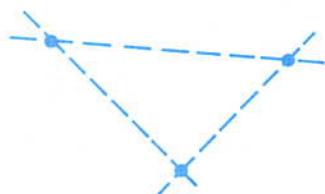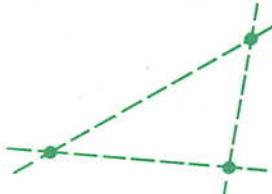
Bill's triangular region   Mary's triangular region

"Now color each corner, and then tear the triangular region into three parts, a corner to each part.
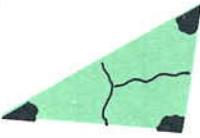
"Next," Mary continued, "I will paste one of my pieces with the colored corner point touching this dot I drew. Then you paste one of your pieces with its colored corner point on the dot I drew — and adjacent to the piece I pasted.
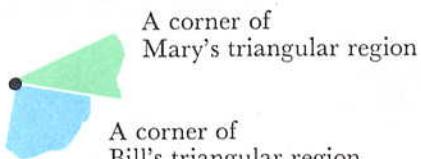
A corner of
Mary's triangular region
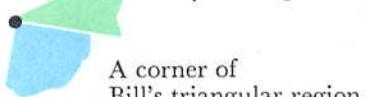
A corner of
Bill's triangular region

"Then I will paste another corner of my triangular region next to yours; you do the same; and I will paste my last corner down. And you paste your last corner down."

"It fits!" Bill exclaimed. "Will wonders never cease!"

"Well," Mary said with a twinkle in her eye, "here is another surprise for you. Start out just as we did before. Draw any shape triangle you like, and so will I. Mark the corners on the triangular region and tear it so each piece contains one corner.

"This time, I'll mark a point on a line and paste one of my corners on one side of the line and you do the same on the other side."

Here is the result:
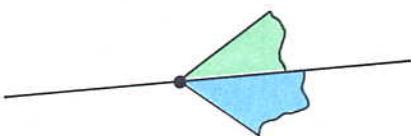
"Then I'll paste a second corner next to my first and you do the same on your side.
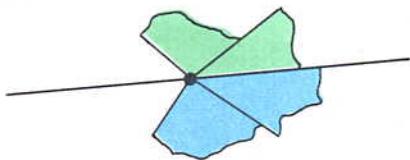
"Finally, I'll paste my last corner down and you do the same," Mary explained.

"Amazing!" Bill said. "Let's try it again, but this time I'm going to change the shape of my triangle."

What was the surprise?

Try it yourself. Cut out two triangular regions of any shape and repeat both of the experiments. Here are some shapes to consider:
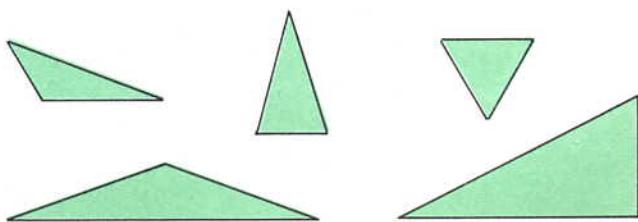
Is there anything in particular to talk about in each of these triangular regions?

50

Bill's Surprises

The next day, Bill came to class with a surprise of his own.

"Mary," he said, "I worked out a trick something like yours. Mark any four points and connect them so you have a four-sided figure. Like this:

"Now, cut along the line that connects the points. Next, tear each shape into four pieces so that each piece contains one of the corners — one of the original four points.
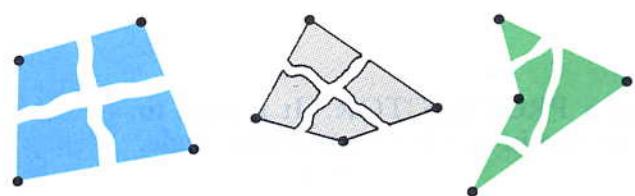
"Then, arrange each group of four pieces so that the corners touch at the same point, and so that their sides just touch.
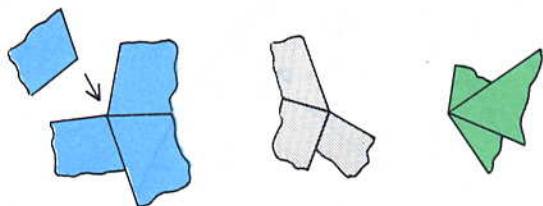
"Surprised?" Bill asked.

"I surely am surprised," Mary replied. "Would it work for squares, rectangles, and other four-sided shapes?"

"Try it!"

"The other fitting trick is this: Cut a triangular corner off a rectangular piece of paper.
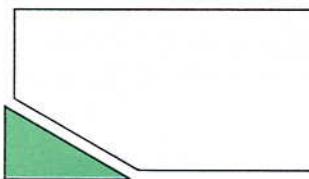
"Tear the triangular corner into three pieces so that each piece has one of the three corners of the triangle. Now lay down the square corner and try to fit the other two corners into it."
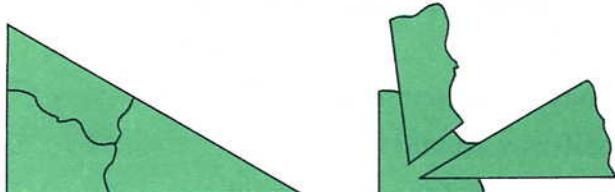
"It's a 'fitting' surprise," Mary agreed. "I wonder if you could get the same results if you tried triangles of different shapes and sizes?"
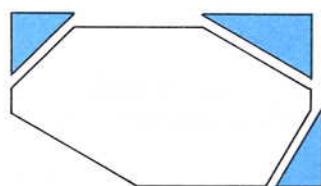
"Try it," Bill said.

Tom had been watching and trying some experiments of his own.

"I think the same thing would happen," he said, "even if you turned all the pieces over or just some of them."

"That's hard to believe," Mary said.

"Try it, then," said Tom.

Make a study of your own of these fitting tricks. Try each trick with many different shapes. Turn some of the pieces over.

As you try each experiment, paste down your results so they become a record of these examples of 'fitting.'

Perhaps you can invent a 'fitting trick' of your own.

Imagine a pointer or clock-hand that can be swung around a pivot. Before it has been swung, suppose it points to the right as shown by the broken line. Of the many positions into which it can turn, a few are shown.

A.   B.   C.   D.   E.   F.
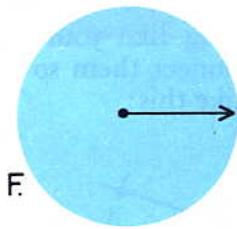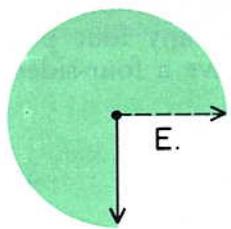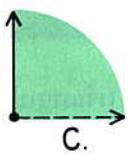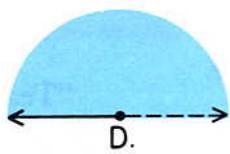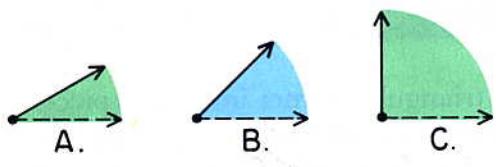
Which example suggests the following amounts of turning?

smallest amount of turning _____

half-way around _____

one-fourth of the way around _____

all the way around _____

three-fourths of the way around _____

half of a quarter of a full turn _____

Mathematicians have agreed to consider a quarter of a full turn as a turn of 90 degrees — in shorthand, 90°.

Then, a half of a turn is _____°.

Three-fourths of a turn is _____°.

A full turn must be _____°.

An instrument such as the one below is called a **PROTRACTOR**. It is used to measure the amount of turning that would be required in making a particular angle.
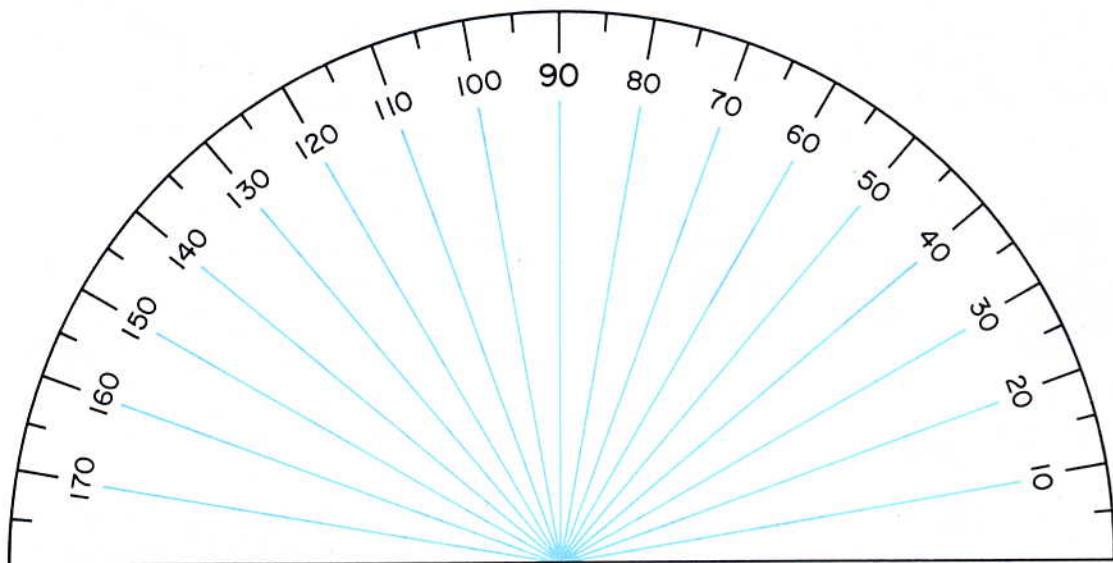
The protractor is a valuable tool. It is used to measure the size of an angle (the amount of turning it would take for a hand to sweep from one side of an angle to the other). Cut out triangles of many different shapes and sizes.

Mark the angles of each a, b, and c. Use the protractor to measure each angle. Record the results in columns III-XIII in the chart on page 53. What do you notice about the sum of the measures of each group of angles?

If the sides of a triangle do not intersect the protractor scale, place a piece of paper along the sides (as the dotted black lines suggest) so that the angles of the triangle can be measured.

a = 80

b = _____

c = _____
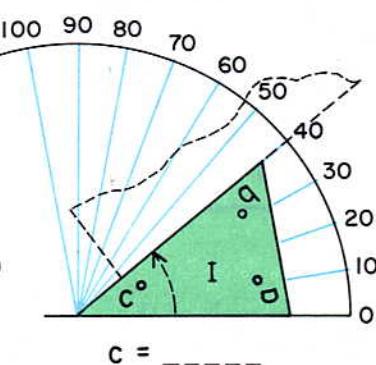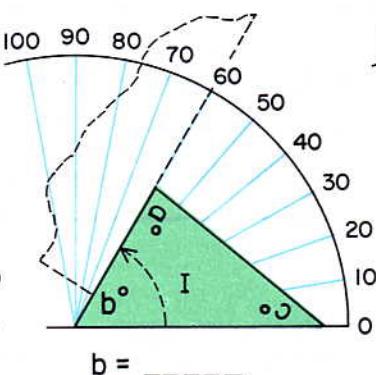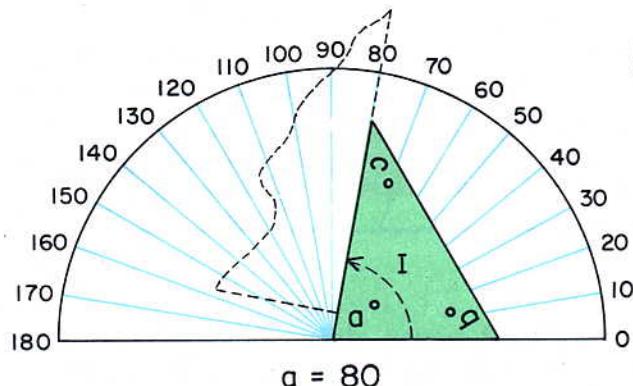
Use the table below to record your experiments with triangles I and II.

Examples:

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII
a	80												
b													
c													
Total													

a = _____

b = _____

c = _____
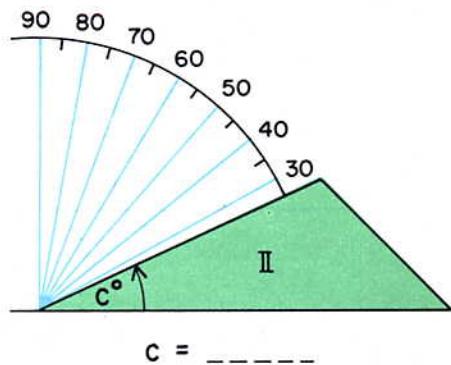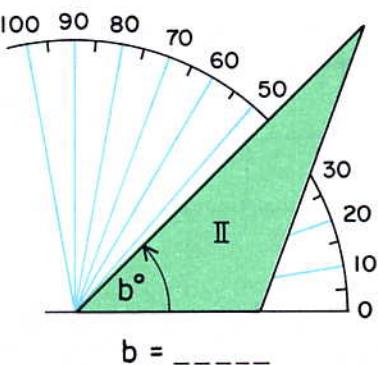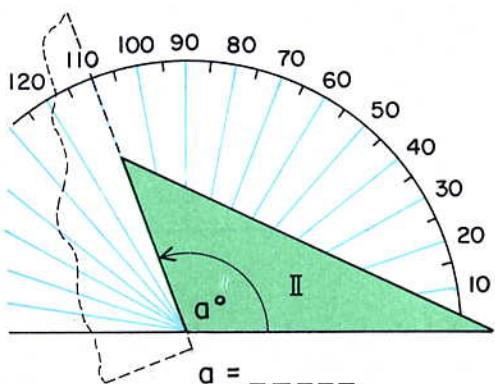
Conduct other experiments. Cut triangular shapes out of paper or cardboard. Place them on the protractor on page 52. Measure each angle and record the results for each triangle in columns III through XIII of the table.

If your measurements could be exact, what would you expect the sum of the degree measures of each group of three angles to be?

Use any triangular shape. Paste one corner down as if you were going to measure the angle, as shown below:

Tear off the other two corners. Fit one as shown. What will happen if the other corner is pushed into place?

53
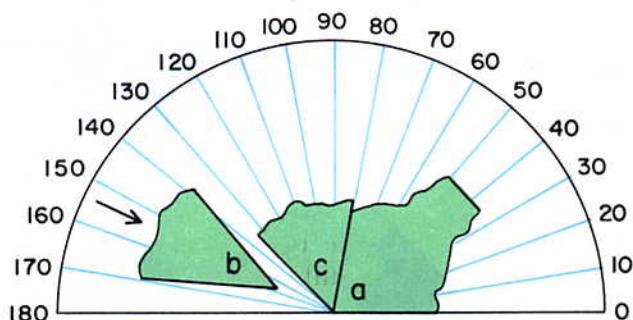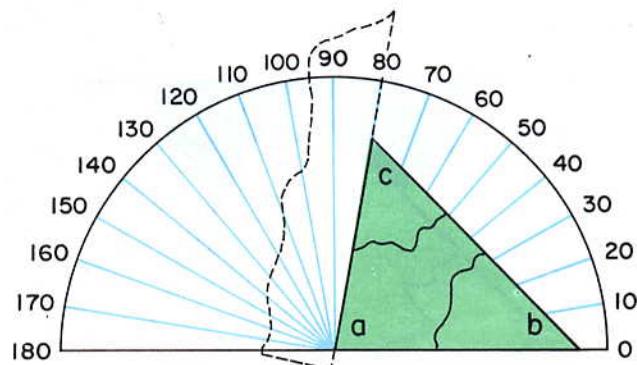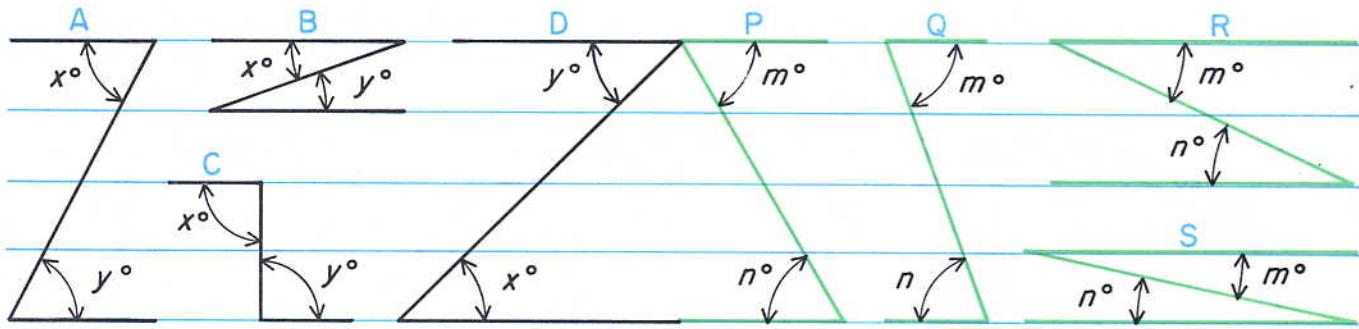
For each of the Z-shapes above, use a protractor to measure angles marked $x°$ and $y°$. For each of the Σ-shapes, measure angles marked $m°$ and $n°$.

On lined paper, draw some Z's of your own — of the regular and reversed varieties — and record similar measurements of the angles in the chart below.

Number of degrees	A	B	C	D	E	F	G	H	I	J	K
x											
y											

Number of degrees	P	Q	R	S	T	U	V	W	X	Y	Z
m											
n											

In shorthand, we can say very briefly what we note: $x = y$ and $m = n$. In the Z and the reverse Z, the angles are the same size.

We call this the "Z-result."

(Turn this page on the side and you might decide to call it the "N-result.")

Below is a copy of the shape or design that appears in the center at the top of the page:
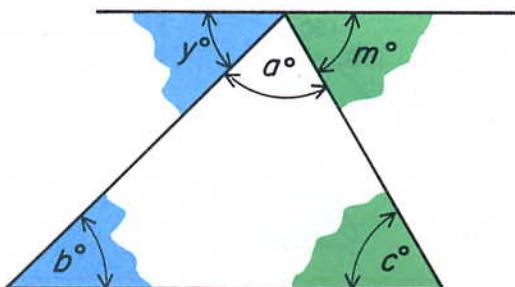
Only the labels have been changed, and a little color has been added.

You can see it as a Z and a Σ put together — or as a triangle with a line through one of the corners.

The Z-result indicates that $b = y$ and that $c = m$.

If we cut out the corners marked $b°$ and $c°$ and paste them on angles marked $y°$ and $m°$, we have:

54
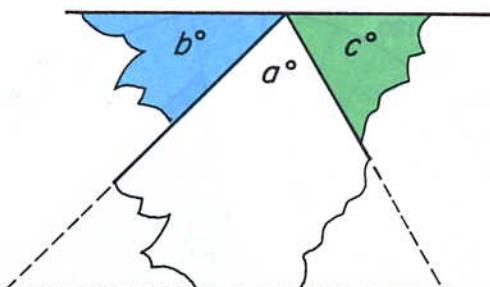
Could you neatly tile a floor using triangular tiles all the same size and shape? We assume that you can cement pieces onto the floor with either side up. Also, you need not worry about the corners or the edges of the room.

Draw a triangular shape on the top sheet of several sheets of paper. Cut as many at a time as possible — so you will have a good supply. (It's more fun if no two angles of your triangular shape are the same size.)

Here is one example:

$a = 80$ $b = 40$ $c =$ ____

> Can you tell without using a protractor?
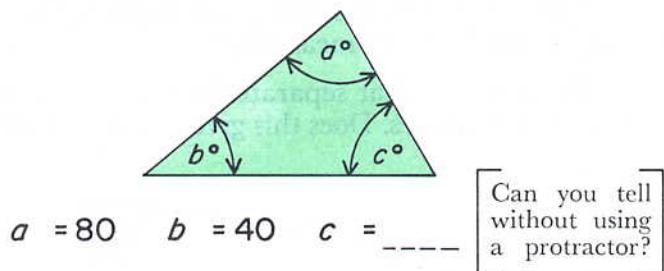
Here is a bridge design:
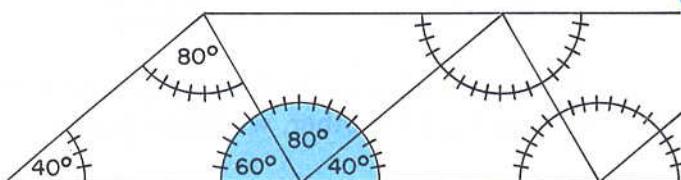
A pattern for a floor of tile the same shape (but smaller) will look like this, some green and some blue.
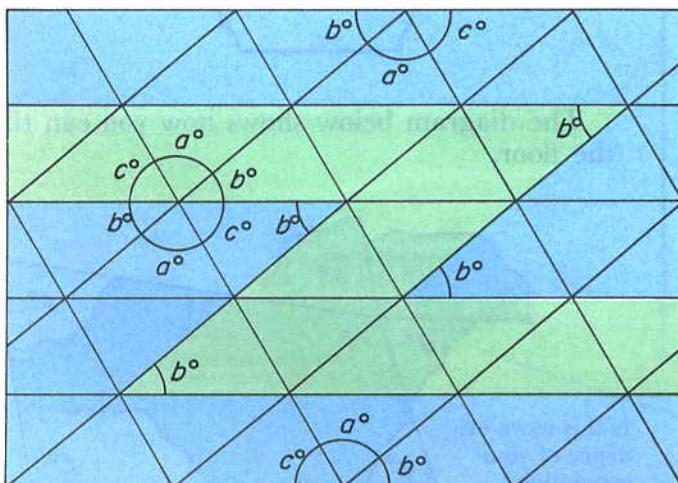
Make some floor covering designs of your own. Paste chunks of these designs on this page. Each time you finish a design, try a different triangular shape. Here are some you might like to choose from:

$m = 60$ $n = 60$

$a = 90$

q is more than 90
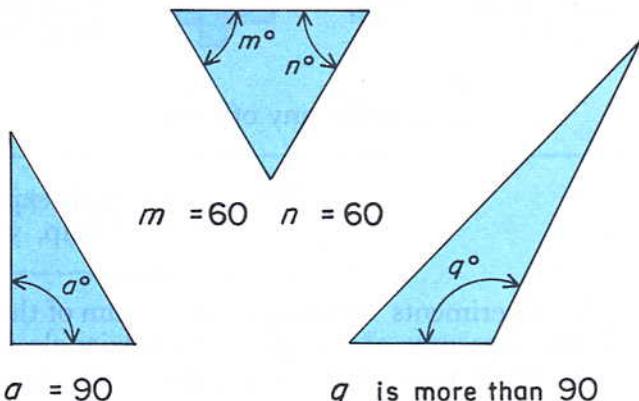
You know that the degree-measure of a quarter of a circle is 90.

If the degree-measure of a quarter of a circle is 90 then a half of a circle measures 2×90 or 180 degrees.

The experiments suggested in these pages have probably led you to the conclusion that the sum of the degree-measures of the angles of a triangle is equal to the degree-measure of a half of a circle.

This conclusion is a most useful fact to remember.
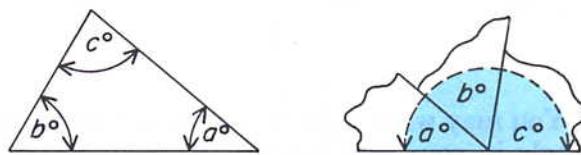
In shorthand, we write: If a triangle has angles of sizes $a°$, $b°$, and $c°$ then

$$a° + b° + c° = 180°$$

55

Experiments with Four-sided Tiles

Of course you can work out many tile patterns with square or rectangular tiles (four-sided tiles).

or □ or ⊞ or ▦
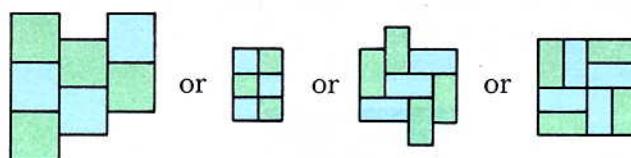
. . . and many others!

It's harder when no two sides of the four-sided tiles are equal in length. Try shapes like these:
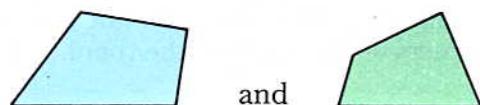
and

Imagine that you have a large supply of tiles shaped like those shown above. Could you cover a floor with such tiles?

NOTE: Try your experiment before going on. If you need help, you may find some hints below.

In experiments we found that the sum of the degree-measures of the angles of a triangle is 180. In a four-sided figure, there are four angles. In the example below, what do you think $q° + r° + s° + t°$ will be?

Measure the angles of several four-sided figures. Keep a record of your measurements.

| | The above example | Your experiments | | | |
		I	II	III	IV
$q°$					
$r°$					
$s°$					
$t°$					
Total					

You may wish to include a square and a rectangle in your experiments. . . .

Please finish this shorthand statement: If a four-sided figure has angles of sizes $q°$, $r°$, $s°$, and $t°$ then

$$q° + r° + s° + t° = \text{_____}$$
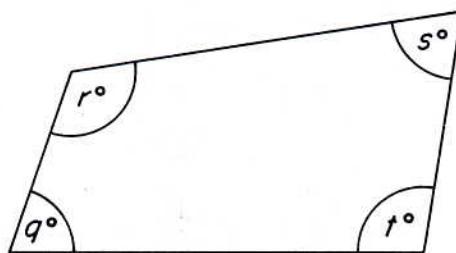
Can you reach the same conclusion by reasoning, without any measuring?

Draw a line that separates the above figure into two triangles. Does this give you any ideas?
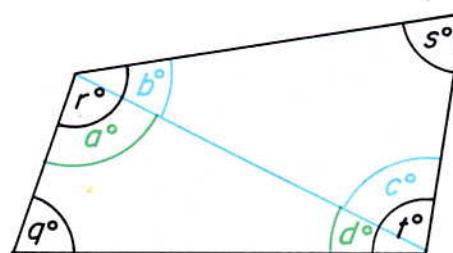
If we were to tear off three corners and fitted them with the other, we might have:
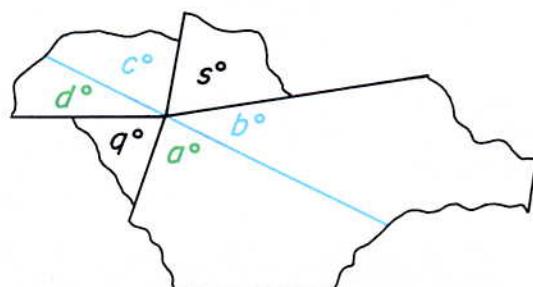
The diagram below shows how you can tile the floor.

Is this more evidence of your conclusion?

56

Alec's Magic Card Machine

Alec left a collection of small machines and a supply of magic cards. Here are sketches of two of his machines and some of his magic cards.
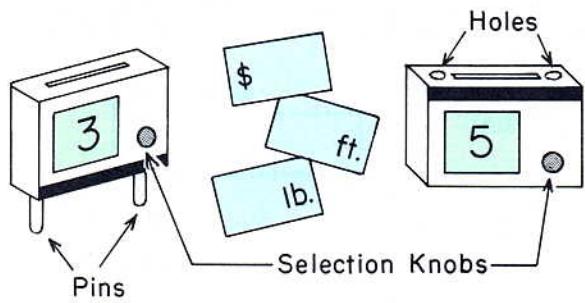
After some experimenting, Mr. Wilson found a way to use the machines. If he wrote an amount on one of the magic cards and put it in the slot on top, another magic card would come out at the bottom.

First he tried the machine with the black bar and the two pins at the bottom. He used the selection knob to show 4 in the green window. On a card with a dollar sign, he wrote 12 . . . and put it through the machine.
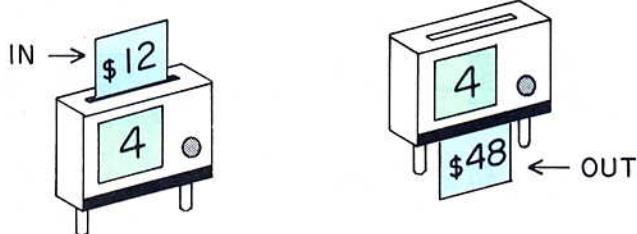
"A surprising machine," Mr. Wilson commented, "but not a surprising result."

Here is a similar experiment. This time the machine has two holes and the black bar at the top.
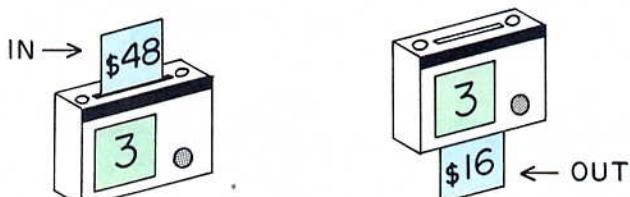
"A few more experiments led me to call one machine a 'Multiplier' and the other a 'Divider.'

"I noticed I could plug one machine into the other (the pins fit the holes). I wrote 10¢ on three cards and put them through the combined machines, turning the selector knobs between each trial. Here is a sketch of the results:
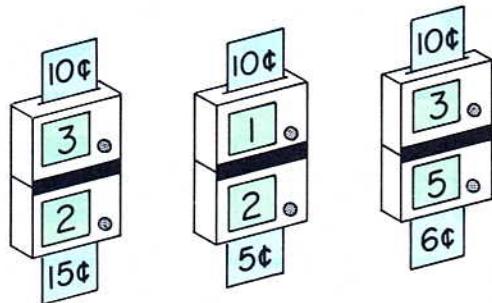
"I thought I knew just what the machines were doing. They were multiplying the 10¢ by the numbers I selected for the top machine and dividing the result by the numbers I selected for the bottom machine. I made notes:

IN	Top Setting		Same		Bottom Setting		OUT
10¢	x 3	=	30¢	and	30¢ ÷	2	= 15¢
10¢	x 1	=	¢	and	¢ ÷	2	= 5¢
10¢	x 3	=	¢	and	¢ ÷	5	= 6¢

"But I was less sure of just what the machines were doing when I reversed the order in which I used the machines. What would happen if I began by dividing 10¢ by the number I selected for the bottom machine? Let's try it.

IN			Same				OUT
10¢ ÷	2	=		and	x 3	=	
10¢ ÷	2	=		and	x 1	=	
10¢ ÷	5	=		and	x 3	=	

"I found a page in Alec's notebook that helped me figure out how Alec used his machines."

When I see a fraction — $\frac{3}{5}$, for example — I think of it as telling me to divide something by 5 and multiply the result by 3. I think of some sketches which will help me to understand the idea better. *(Please fill in any missing instructions.)*
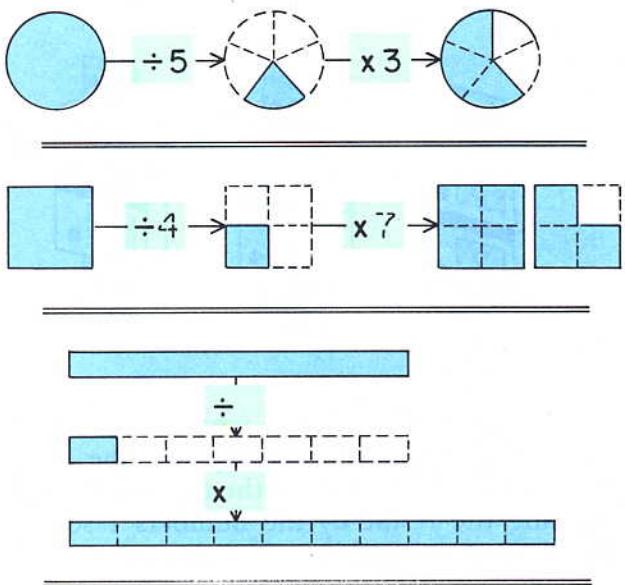
Special kinds of sketches illustrate the idea that different pairs of instructions may lead to the same final result.
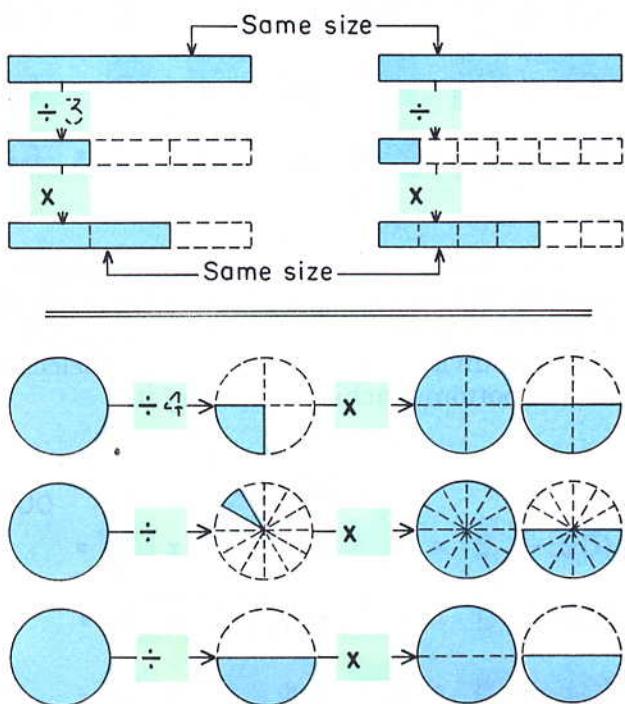
(You may wish to draw sketches of your own to show that different pairs of instructions may lead to the same final result.)

Sketches can also help to convince me that the order of carrying out a pair of instructions does not affect the final result.

(Please fill in the instructions.)
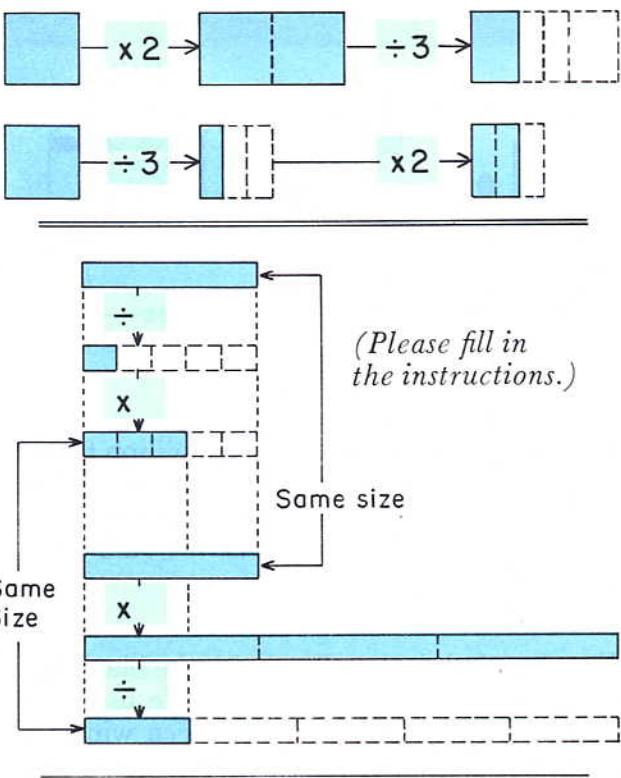
I can simplify my sketches by doing the following:
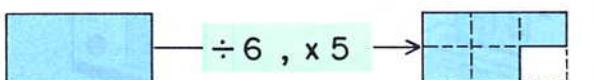
(Please fill in missing instructions.)
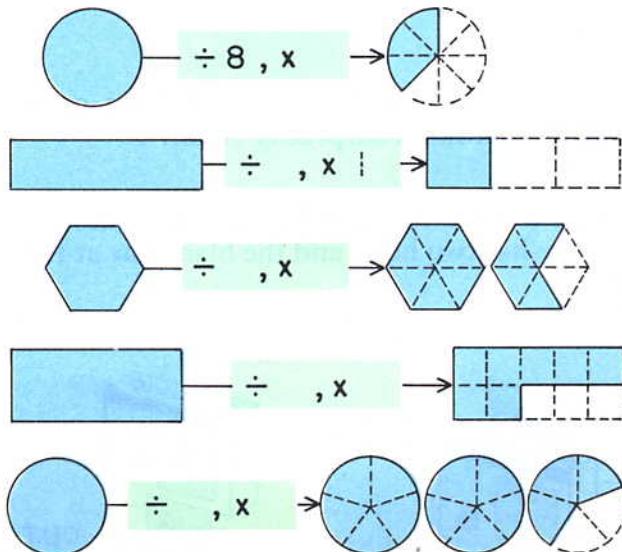
Mr. Wilson experiments
with Alec's
"Fraction Machine"

```
┌─────┐
│  2 ●│
├─────┤
│  5 ●│
└─────┘
```
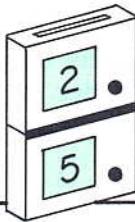
"As I began working with Alec's machine," Mr. Wilson told the class, "I developed a short-hand to keep track of my results.

"In the beginning, I used a little multiplication sign next to my record of the top machine-setting and a little division sign next to my record of the bottom machine-setting.

"Here are a few examples."

Please supply the missing entries.

IN →	14¢	8 ft.	$10	15 oz.
	x 1	x 3	x 9	x 7
	÷ 2	÷ 4	÷ 5	÷ 3
OUT →	7 ¢	ft.	$	oz.

"I soon learned that frequently the arithmetic was easier if I carried out the division first and then multiplied.

"Alec's notes suggested that several settings of the selectors could produce the same final results. My experiments showed he was right."

30¢	30¢	30¢	30¢
x 4	x	x	x 6
÷ 6	÷15	÷ 3	÷
20¢	20¢	20¢	20¢

"That last example reminded me that sometimes it is more convenient to multiply first and then divide. 30¢ ÷ 9 is not a whole number of cents; but 180¢ ÷ 9 = 20¢."

Mr. Wilson gave the class some practice in deciding which pairs of machine-settings produced the same results. He gave an input card and an output card. Then he wrote out several pairs of machine-settings. He explained:

"Check (✓) machine-settings that would produce the results shown; cross out (✗) those that would not. Then write the machine-setting with the smallest numbers in the boxes on the left."

12 oz.	✓			
x	x 2̶ ✗ ÷3̶	x 2	x 1	x 3
		÷ 4	÷ 2	÷ 4
÷	x 6	x 3	x 12	x 5
6 oz.	÷ 1	÷ 6	÷ 6	÷ 10

24 in.	✓			
x	x 3	x 2	x 12	x 9
	÷ 2	÷ 3	÷ 8	÷ 6
÷	x 21	x 6	x 3	x 15
36 in.	÷ 14	÷ 4	÷ 12	÷ 10

$60				
x	x 18	x 6	x 10	x 15
	÷ 30	÷ 10	÷ 15	÷ 25
÷	x 9	x 16	x 20	x 12
$36	÷ 15	÷ 3	÷ 12	÷ 20

72¢				
x	x 16	x 7	x 54	x 42
	÷ 28	÷ 4	÷ 35	÷ 24
÷	x 14	x 63	x 21	x 8
126¢	÷ 8	÷ 36	÷ 12	÷ 14

In the pages that follow, Mr. Wilson carries out many experiments with Alec's "Fraction Machine."

Many of his entries are missing. Please fill them in.

Perhaps you might make up experiments of your own.

Another Investigation

"Often when I was experimenting with Alec's machines, I would use the same input with many different settings on the multiplier and divider machines (in green)."

x1 ÷2	x1 ÷3	x2 ÷3	x3 ÷3	x1 ÷4	x2 ÷4	x3 ÷4	x4 ÷4	x1 ÷6	x2 ÷6	x3 ÷6
6	4									

12 EGGS

All numbers written in blue blocks are numbers of eggs.

x4 ÷6	x5 ÷6	x6 ÷6	x1 ÷12	x2 ÷12	x3 ÷12	x4 ÷12	x5 ÷12	x6 ÷12	x7 ÷12	x8 ÷12	x9 ÷12	x10 ÷12	x11 ÷12

x12 ÷12	x2 ÷2	x3 ÷2	x4 ÷3	x5 ÷4	x6 ÷4	x4 ÷2	x8 ÷6	x6 ÷2	x9 ÷6	x8 ÷4	x7 ÷3	x15 ÷12	x5 ÷2

"Here are some machine-settings that gave me results."

6	x1 ÷2	x2 ÷4	x3 ÷	x6 ÷

4	x ÷3	x2 ÷	x4 ÷

8	x2 ÷	x ÷	x ÷

3	x ÷	x ÷

9	x ÷	x ÷

2	x ÷	x ÷

10	x ÷	x ÷

12	x ÷	x ÷	x ÷	x ÷	x ÷

18	x ÷	x ÷	x ÷

24	x ÷	x ÷

---

one dime

x1 ÷10	x1 ÷5	x1 ÷2	x2 ÷10	x2 ÷5	x2 ÷2	x3 ÷10	x3 ÷5	x3 ÷2	x4 ÷10	x4 ÷5

10 CENTS

All numbers written in blue blocks are numbers of cents.

x4 ÷2	x5 ÷10	x5 ÷5	x10 ÷20	x6 ÷10	x15 ÷10	x1 ÷1	x7 ÷10	x6 ÷15	x2 ÷2	x9 ÷15	x5 ÷5	x12 ÷20	x10 ÷10

5	x1 ÷2	x ÷	x ÷

4	x ÷	x ÷	x ÷

6	x ÷	x ÷	x ÷	x ÷

"Since Alec always put the multiplier machines on top and divider machines on the bottom, I began to leave out the ' × ' signs and the ' ÷ ' signs."

$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{7}{5}$
20													

40 minutes

Numbers written in blue blocks are numbers of minutes.

$\frac{1}{8}$	$\frac{2}{8}$	$\frac{4}{8}$	$\frac{6}{8}$	$\frac{8}{8}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{5}{10}$	$\frac{5}{20}$	$\frac{10}{20}$	$\frac{30}{20}$	$\frac{2}{20}$	$\frac{12}{8}$	$\frac{15}{10}$	$\frac{1}{40}$	$\frac{4}{40}$	$\frac{50}{40}$	$\frac{60}{40}$

Groups of machine-settings that give the same results.

20	—	—	—	—	—

30	—	—

4	—	—	—

10	—	—	—

40	—	—	—

60	—	—	—	—	—	—

$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{2}{1}$	$\frac{5}{2}$	$\frac{6}{4}$	$\frac{2}{8}$	$\frac{10}{4}$	$\frac{3}{1}$	$\frac{2}{4}$	$\frac{20}{8}$	$\frac{6}{8}$	$\frac{4}{2}$
8													

one pound

16 ounces

Numbers written in blue blocks are numbers of ounces.

$\frac{10}{2}$	$\frac{8}{4}$	$\frac{4}{16}$	$\frac{12}{4}$	$\frac{24}{16}$	$\frac{7}{4}$	$\frac{4}{8}$	$\frac{40}{16}$	$\frac{16}{8}$	$\frac{12}{8}$	$\frac{40}{8}$	$\frac{8}{16}$	$\frac{24}{8}$	$\frac{32}{16}$	$\frac{14}{8}$	$\frac{12}{16}$	$\frac{6}{2}$	$\frac{28}{16}$

Groups of machine-settings that give the same results.

8	—	—	—	—

4	—	—	—

24	—	—	—	—

12	—	—	—

32	—	—	—	—	—

40	—	—	—

48	—	—	—	—

28	—	—	—

80	—	—	—

"In the following records," Mr. Wilson explained, "I find several pairs of machine-settings that would produce the same result.

"I make sure that one pair of the machine-settings uses the smallest whole numbers possible. I mark this pair of machine-settings with a check (✓)."

Please complete these records.

NOT A LEAP YEAR!

FEBRUARY
S	M	T	W	T	F	S
				1	2	3	4	5
6	7	8	9	10	11	12
13	14					

28 days

✓ $\frac{2}{1}$ $\frac{\_}{2}$ $\frac{\_}{4}$ $\frac{10}{\_}$ = 56 56 56 56

✓ $\frac{3}{\_}$ $\frac{\_}{28}$ $\frac{6}{\_}$ = 21 21 21

✓ $\frac{\_}{14}$ $\frac{\_}{\_}$ $\frac{\_}{28}$ = 14 14 14

✓ $\frac{\_}{\_}$ $\frac{\_}{\_}$ = 7 7

Numbers written in blue blocks are numbers of days.

✓ $\frac{14}{\_}$ $\frac{\_}{\_}$ $\frac{\_}{\_}$ = 8 8 8

✓ $\frac{\_}{\_}$ $\frac{6}{\_}$ $\frac{\_}{\_}$ = 12 12 12

✓ $\frac{28}{\_}$ $\frac{\_}{\_}$ $\frac{\_}{\_}$ = 20 20 20

✓ $\frac{\_}{\_}$ $\frac{2}{\_}$ $\frac{\_}{\_}$ = 28 28 28

✓ $\frac{\_}{\_}$ $\frac{\_}{\_}$ $\frac{4}{\_}$ $\frac{\_}{\_}$ = 84 84 84 84

---

foot ruler — 6

9 inches

✓ $\frac{\_}{\_}$ $\frac{\_}{9}$ $\frac{6}{\_}$ = 3 3 3

✓ $\frac{\_}{\_}$ $\frac{6}{\_}$ $\frac{\_}{18}$ = 6 6 6

✓ $\frac{\_}{\_}$ $\frac{8}{\_}$ $\frac{\_}{36}$ = 4 4 4

✓ $\frac{\_}{\_}$ $\frac{\_}{9}$ $\frac{8}{\_}$ = 12 12 12

Numbers written in blue blocks are numbers of inches.

✓ $\frac{\_}{\_}$ $\frac{4}{\_}$ $\frac{10}{\_}$ $\frac{\_}{3}$ = 18 18 18 18

✓ $\frac{\_}{\_}$ $\frac{\_}{\_}$ $\frac{\_}{\_}$ $\frac{\_}{\_}$ = 27 27 27 27

✓ $\frac{\_}{\_}$ $\frac{\_}{\_}$ $\frac{\_}{\_}$ $\frac{\_}{\_}$ = 15 15 15 15

✓ $\frac{\_}{\_}$ $\frac{\_}{\_}$ $\frac{\_}{\_}$ $\frac{\_}{\_}$ = 5 5 5 5

DIME DIME DIME

30 cents

Numbers written in blue blocks are numbers of cents.

✓ $\frac{\_}{\_}$ $\frac{\_}{6}$ $\frac{\_}{30}$ $\frac{5}{\_}$ $\frac{4}{\_}$ $\frac{3}{\_}$ $\frac{6}{\_}$ = 10 10 10 10 10 10 10

✓ $\frac{\_}{\_}$ $\frac{\_}{\_}$ $\frac{\_}{10}$ $\frac{\_}{30}$ $\frac{6}{\_}$ $\frac{8}{\_}$ $\frac{10}{\_}$ = 12 12 12 12 12 12

✓ $\frac{\_}{\_}$ $\frac{\_}{6}$ $\frac{15}{10}$ $\frac{2}{\_}$ $\frac{\_}{\_}$ $\frac{\_}{\_}$ $\frac{\_}{\_}$ = 15 15 15 15 15 15 15

✓ $\frac{\_}{\_}$ $\frac{\_}{6}$ $\frac{\_}{\_}$ $\frac{6}{\_}$ $\frac{\_}{\_}$ = 20 20 20 20 20 20

✓ $\frac{\_}{\_}$ $\frac{30}{\_}$ $\frac{\_}{\_}$ $\frac{\_}{6}$ = 30 30 30 30

✓ $\frac{\_}{\_}$ $\frac{\_}{\_}$ $\frac{\_}{\_}$ $\frac{5}{\_}$ = 3 3 3 3 3

✓ $\frac{\_}{\_}$ $\frac{\_}{\_}$ $\frac{45}{\_}$ $\frac{\_}{\_}$ = 45 45 45 45

✓ $\frac{\_}{\_}$ $\frac{\_}{\_}$ $\frac{\_}{3}$ $\frac{8}{\_}$ $\frac{\_}{\_}$ = 60 60 60 60 60

✓ $\frac{\_}{\_}$ $\frac{22}{\_}$ = 11 11

"Have you noticed that the kind of units can change while the number of units remains the same?"

"Please complete these records," Mr. Wilson asked.

24 inches yardstick

24 tiles

24 eggs

24 hours

1 lb. ½ lb

10¢ 10¢ 24 cents
1¢ 1¢ 1¢ 1¢

24

✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
—	—	—	—	—	—	—	—	—	—
6	8	4	24	12	18	3	10	9	20

Find several pairs of machine-settings that produce the same output. Include the pair of machine-settings that uses the smallest possible whole numbers, and record it in the box with the check mark (✓).

Row 1 (with check marks):

		✓				✓		16	✓	10		✓			✓
—	—	—	—	—	—	—	—	6	—	—	16	—	—	36	36
8	8	8	2	2	2	48	48	48	16	16	16	30	30	30	

Row 2:

✓	✓	✓	✓	14	✓	3	✓	50	✓	26	✓
—	—	—	—	—	—	—	—	—	—	—	—
7	7	15	15	21	21	24	27	27	14	14	1	1	25	25	13	13

5 Dozen Eggs
60 eggs

6 Dimes
60 cents

1 Hour
60 minutes

60

6 Dimes 60 cents; 1 Hour 60 minutes

Row:
4	✓	12	✓	20	✓	2	✓	6	✓
30	30	45	45	15	15	20	20	40	40

20 ✓	30 ✓	5 ✓	4	60 ✓	7 ✓	6 ✓	6 ✓
12	12	18	18	10	10	24	24	50	50	60	60	90	90	120	120

$1 one dollar
100 cents
100 miles
100 apples
100 minutes
100 pounds
100 feet

100

Row:
20 ✓	10 ✓	10 ✓	60 ✓	3 ✓
25	25	50	50	10	10	5	5	1	1

20 ✓	60 ✓	45 ✓	6	21 ✓	10 ✓	115 ✓	10 ✓
75	75	70	70	45	45	150	150	7	7	200	200	23	23	80	80

63

Mr. Wilson describes a more complicated

FRACTION MACHINE

designed by Alec Marson.

"This strange machine has 5 selector knobs. One to select a number for each of the four green windows, and another that selects '+,' '−,' '×,' or '÷' for the center indicator.

"The machine has 2 pins on the right."

Samples of Mr. Wilson's records
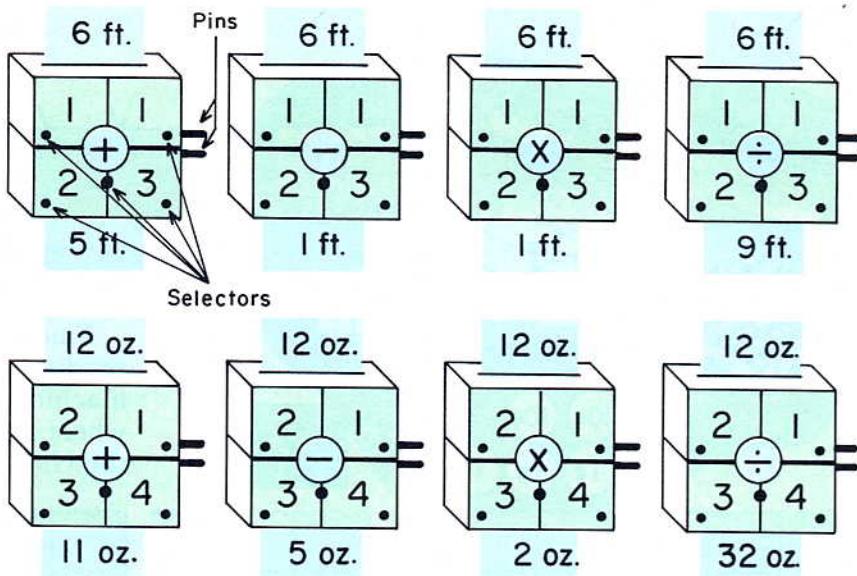
Have you any hunches as to how this new machine yields these outputs?

Mr. Wilson has a few clues.

"I found that certain combinations of machine-settings gave surprising results. The outputs are just the same as I would get from the other, simpler machine set according to the left-hand setting on this machine. Here are some results in a kind of shorthand."

(Please fill in the missing entries.)

A surprise!

"I certainly did not expect those last two outputs. What happened?

"In my next experiments, I reversed the machine-settings on the right with those on the left."

"Another surprise. In examples set with '+' and '×,' reversing the machine-settings from left to right did not affect the output. So I went back to the first pattern of machine-settings."

(Please fill in the missing entries.)

Mr. Wilson described his next series of experiments.

"I compared the settings on the new machine and on pairs of the simple machines . . . using the ' + ' and ' − ' center settings."

(Please fill in the missing entries.)

New machine | A pair of hookups of old machines

21¢
```
| 4 | 1 |
|  (+)  |
| 7 | 3 |
```
19¢

21¢
```
| x 4 |
| ÷ 7 |
```
¢

21¢
```
| x 1 |
| ÷ 3 |
```
¢

$ 36
```
| 3 | 1 |
|  (−)  |
| 4 | 9 |
```
$ 23

$ 36
```
| x 3 |
| ÷ 4 |
```
$

$ 36
```
| x 1 |
| ÷ 9 |
```
$

"Did you notice anything about those results?

"I did, and I kept records of the next experiments in this way."

Addition

12¢
```
| 3 | 1 |
|  (+)  |
| 4 | 6 |
```
9¢ + ¢
11¢

$ 9
```
| 2 | 4 |
|  (+)  |
| 3 | 2 |
```
$ + $
$

40 oz.
```
|   | 1 |
|  (+)  |
| 5 |   |
```
24 oz.+ 10 oz.
oz.

Subtraction

$ 18
```
| 2 | 1 |
|  (−)  |
| 3 | 2 |
```
$ − $ 9
$ 3

36¢
```
| 5 | 2 |
|  (−)  |
| 6 | 9 |
```
¢ − ¢
¢

30 ft.
```
| 2 | 3 |
|  (−)  |
|   |   |
```
20 ft. − 18 ft.
ft.

"I also tried the same approach with the ' × ' and ' ÷ ' settings. I'll discuss my results later."

"Now I could know ahead of time the results of Alec's machines when they were set for addition or subtraction.

"Here are some of my notes."

(Please fill in the missing entries.)

$ 72
```
| 5 | 2 |
|  (+)  |
| 8 | 3 |
```
$

45 ft.
```
| 2 | 1 |
|  (+)  |
|   | 3 |
```
33 ft.

60¢
```
| 5 | 3 |
|  (−)  |
| 6 | 4 |
```
¢

20 oz.
```
| 4 |   |
|  (−)  |
| 1 | 4 |
```
45 oz.

54 lb.
```
| 8 | 7 |
|  (−)  |
|   | 3 |
```
18 lb.

$ 70
```
| 2 | 3 |
|  (+)  |
|   | 7 |
```
$ 50

51 hr.
```
| 2 |   |
|  (+)  |
| 3 | 1 |
```
34 hr.

60 in.
```
| 9 | 7 |
|  (−)  |
| 5 | 12 |
```
in.

"I still needed to find why the machine rejected one of my earlier experiments.

"Why?"

12 eggs
```
| 0 | 2 |
|  (−)  |
| 1 | 3 |
```
REJECT

"Here are other examples in which the output was 'REJECT.' "

36¢
```
| 2 | 5 |
|  (+)  |
| 9 | 8 |
```
REJECT

$ 10
```
| 20 | 2 |
|  (−)  |
| 5 | 3 |
```
REJECT

20 in.
```
| 1 | 1 |
|  (−)  |
| 5 | 4 |
```
REJECT

21 lb.
```
| 3 | 2 |
|  (−)  |
| 7 | 3 |
```
REJECT

"Some were rejected for one reason and some for another.

"It seems that the dividing mechanism in Alec's machines can handle only counting numbers — it is not equipped to respond to '36¢ ÷ 8' nor to '4 − 5.'

"Now I'm ready to tackle the problem of those ' × ' and ' ÷ ' machine-settings."

More Experiments with Alec's Fraction Machine

"I tried comparing the output of Alec's new machine set at '×' with a pair of Alec's simpler machine hookups . . . just as I had with the new machine set at '+' and '−.' For a long time, I found no clues.

"Then, almost by accident, I found a most surprising result. I kept a record of it.

30¢	
1	2
⊗	
3	5
4¢	

30¢	→	10¢
x 1		x 2
÷ 3		÷ 5
10¢		¢
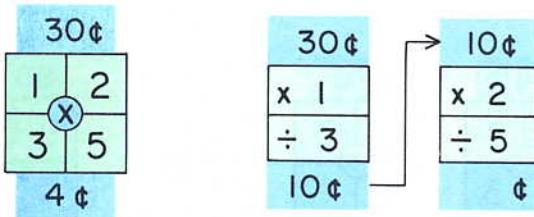
"I took the output from the first simple fraction machine and used it as an input for a second one.

"So I tried again."   (Please fill the blanks.)

$ 36	
7	2
⊗	
4	3
$ 42	

$ 36	→	$
x 7		x 2
÷ 4		÷ 3
$		$
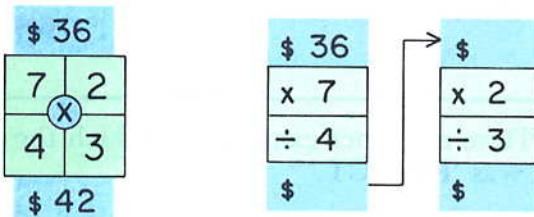
"As I made more experiments, I kept records in this way:

45¢	90 ¢
2	4
⊗	
1	9
90 ¢	¢

48¢	12 ¢
1	5
⊗	
4	6
¢	¢

100 in.	in.
1	1
⊗	
2	2
in.	in.

72 hr.	hr.
3	5
⊗	
4	9
hr.	hr.
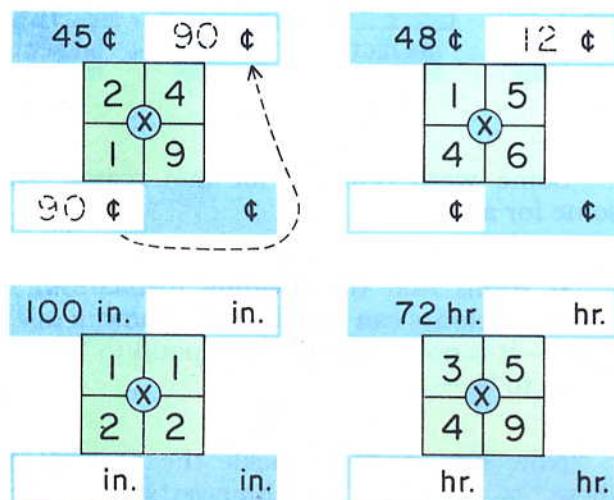
"Then, for a few experiments, I kept my records this way:

100 lb.	(20 lb.)
1	1
⊗	
5	5
(20 lb.)	lb.

4 lb.	(lb.
5	5
⊗	
1	1
(lb.)	lb.

$ 42	($)
2	1
⊗	
7	3
($)	$

$ 42	($)
1	2
⊗	
3	7
($)	$
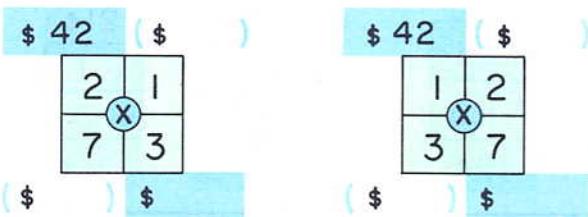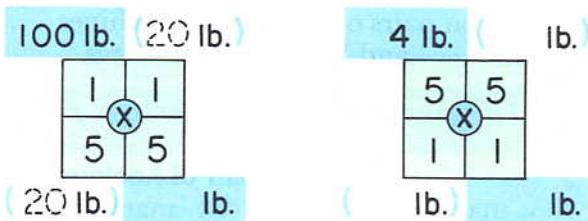
"Finally, I left out the record of the first output that was also the input for the second step. This enabled me to keep more records in less space."

(Please fill the blanks.)

12 oz.		12 ft.		$ 54		54 hr.	
1	1	2	3	7	4	4	7
⊗		⊗		⊗		⊗	
2	3	1	1	3	9	3	9
oz.		ft.		$		hr.	

50¢		50 lb.		$ 72		72 yd.	
2	6	2	6	3	8	3	8
⊗		⊗		⊗		⊗	
5	10	10	5	3	8	8	3
¢		lb.		$		yd.	
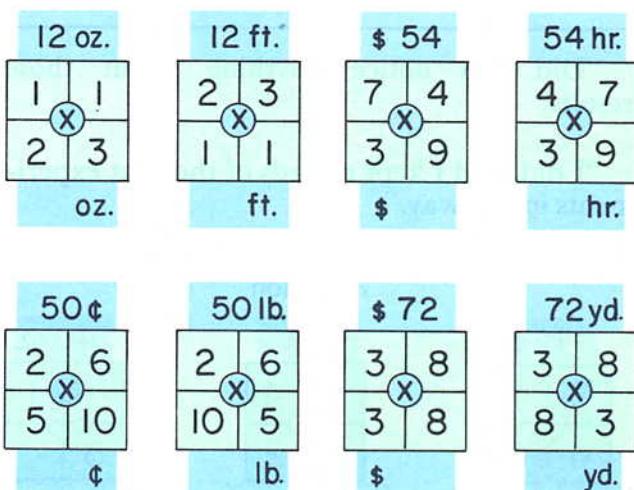
"I wondered what would happen if I used a zero in each group of settings.

12 min.		72¢		8 hr.		$ 25	
0	5	7	0	2	5	4	6
⊗		⊗		⊗		⊗	
2	3	9	4	0	4	5	0
0 min.		¢		REJECT		REJECT	
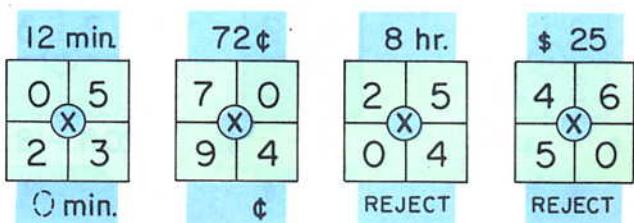
"I should have expected those rejects — it makes no sense to divide by 0."

"The key clue which led to my discovering the secret of Alec's fraction machine with a ' ÷ ' setting was the fact that we cannot find any reasonable meaning for division by zero.

"Sometimes, people want to say that dividing a measurement by zero does not change the measurement. That leads to a trap:

If 21¢ ÷ 3 = 7¢   then   7¢ x 3 = 21¢

If 21¢ ÷ 0 = 21¢   then 21¢ x 0 = 21¢

and that is not true!

"Sometimes, people want to say that any amount divided by zero is 'infinity' — ∞ in shorthand. Whatever they mean by 'infinity,' they fall into this trap:

25 miles ÷ 0 = ∞ and   0 x ∞ = 25 miles

9 miles ÷ 0 = ∞ and   0 x ∞ = 9 miles

so 25 miles = 9 miles. (False!)

"Mathematicians have never discovered a reasonable meaning for division by zero. So they reject the idea . . . as Alec's machine does."

Reversing the instructions!

"I tried the experiment of using the output of one of Alec's machines as input for another — as I had with the ' X ' settings:
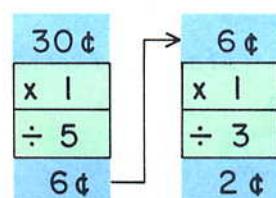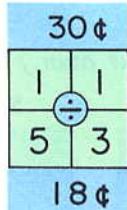
"If Alec's second machine had only multi-

plied by 3 instead of divided by 3 and divided by 1 instead of multiplied by 1!

"The fraction machine with a ' ÷ ' setting acts just like a fraction machine with a 'X' setting but with the numbers on the right interchanged.
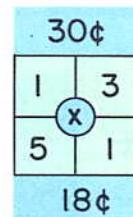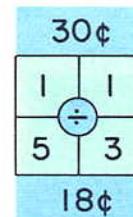
"Maybe I should try switching the 'X' and '÷' on Alec's second machine hookup. I tried a new example involving the fraction machine with a '÷' setting:
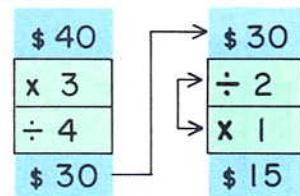
— Ah! . . . the same —

"I tried another example:
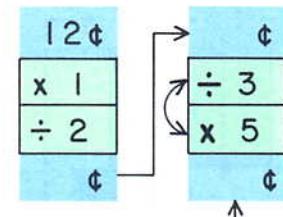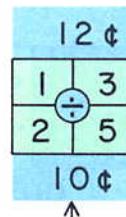
"Try again!"
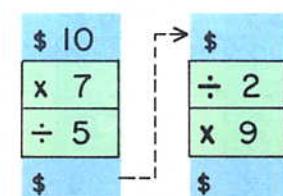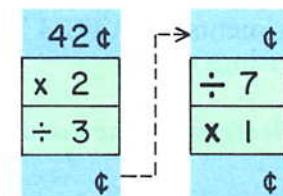
"Here are records of more experiments with the '÷' setting on Alec's fraction machine." *(Please fill in the missing entries.)*

1.

9 ft.	6 ft.
x → 2	2 ← ÷
÷ → 3	3 ← x
6 ft.	___ ft.

2.

1 oz.	___ oz.
12	4 ← ÷
1	1 ← x
___ oz.	___ oz.

3.

18 in.	___ in.
	3 ← ÷
2	1 ← x
___ in.	3 in.

4.

___ doz.	___ doz.
2	1 ← ÷
5	4
___ doz.	24 doz.

5. 20 lb.

3	5 ← ÷
2	1

___ lb.

6. 12 yd.

5	3
4	1

___ yd.

7. 50¢

1	5
2	2

___ ¢

8. $ 6

2	4
3	5

___ $

9. 14 gal.

3	3
7	11

___ gal.

10. 3 qt.

20	4
1	1

___ qt.

11. 24 pt.

1	1
6	3

___ pt.

12. $ 36

2	0
9	4

___

"I made some sketches to help me remember the way Alec's machines work.
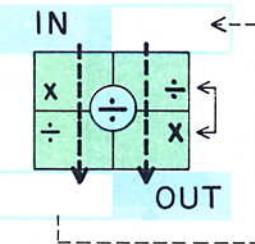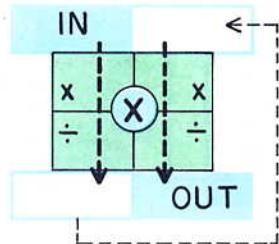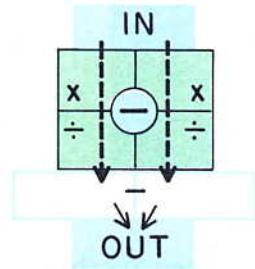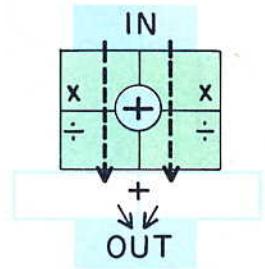
"Here are samples from the records of my experiments. WATCH THE SIGNS (the center settings)."

(Please fill in the missing entries.)

13. 56¢

3	1
(+)	
4	2

___ ¢

14. 48¢

3	1
(−)	
4	2

___ ¢

15. 80¢

3	1
(x)	
4	2

___ ¢

16. 60¢

3	1 ← ÷
4	2 ← x

___ ¢

17. 7 ft.

3	5
(+)	
1	1

___ ft.

18. 10 ft.

9	4
(−)	
1	1

___ ft.

19. 3 ft.

2	10
(x)	
1	1

___ ft.

20. 7 ft.

8	4
(÷)	
1	1

___ ft.

"In the following examples, use any inputs that Alec's fraction machine will not RE-JECT."

21.

1	2
(+)	
4	3

22.

3	1
(−)	
5	5

23.

1	1
(x)	
2	2

24.

2	2
(÷)	
5	5

25.

7	1
(+)	
1	2

26.

3	1
(−)	
1	2

27.

2	3
(x)	
3	4

28.

2	4
(÷)	
3	3

(Make up examples of your own.)

29.

(+)	

30.

(−)	

31.

(x)	

32.

(÷)	

A.
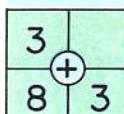
```
        24 bks.
       ┌───┬───┐
       │ 3 │   │
       │  (+)  │
       │ 8 │ 3 │
       └───┴───┘
        bks.  bks.
          bks.
```

Alice and Mary had agreed to collect twenty-four used books for the school book sale. Alice had collected three-eighths of that number of books. Mary had collected one-third of that number of books.

Alice had collected _____ books. Mary had collected _____ books. Together, they had collected _____ books. The number of books they both collected is _____ less than their goal.

B.
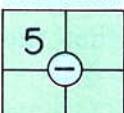
```
        60 min.
       ┌───┬───┐
       │ 5 │   │
       │  (−)  │
       │   │   │
       └───┴───┘
        min.  min.
          min.
```

One day Terry spent five-sixths of an hour on his homework and practiced playing his violin for three-fifths of an hour.

He spent _____ minutes more on his homework than on his violin practice.

C.
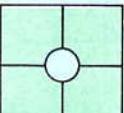
```
        36 in.
       ┌───┬───┐
       │   │   │
       │  (○)  │
       │   │   │
       └───┴───┘
        in.   in.
          in.
```

Five-sixths of a yard added to three-fourths of a yard is _____ inches; that is _____ (more or less) than five feet.

D.
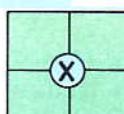
```
     300 tks.      tks.
       ┌───┬───┐
       │   │   │
       │  (×)  │
       │   │   │
       └───┴───┘
        tks.   tks.
```

The Drama Club had 300 tickets printed. Three-fifths of those printed were sold. Of those sold, three-fourths were adult tickets.

There were _____ adult tickets sold.

E.

```
     $ 12.00   $
       ┌───┬───┐
       │ 5 │   │
       │  (×)  │
       │   │   │
       └───┴───┘
         $     $
```
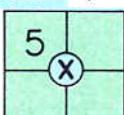
A group of four boys worked for five days. As a group, they received $12.00 a day. The boys divided their money evenly.

Jack spent half of his earnings. He must have spent $_____.

If Jack had spent a third of his earnings instead of a half, he would have spent _____ (more or less) than he actually spent.

F.

```
     $ 36     $
       ┌───┬───┐
       │ 1 │ 3 │
       │  (÷)  │
       │ 4 │   │
       └───┴───┘
         $     $
```
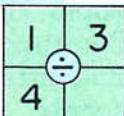
Bill earned thirty-six dollars. He put all but one-fourth of it in the bank. With the amount he had left, he could buy exactly three of the equally priced model rockets he wanted. He bought two models.

Bill spent a total of $_____ for the two rockets.

WHAT CAN YOU SAY?

1. Boyd had less than a quarter. He spent a fifth of his money for gum and three-fourths of his money for ice cream.

What can you say?

If Boyd had 12¢ then he spent _____¢ for ice cream.

If he had 15¢ then he spent _____¢ for gum.

Boyd spent _____ (more or less) for gum than he did for ice cream.

If Boyd spent 18¢ for ice cream then he had _____¢.

You can be sure that Boyd spent _____¢ for gum, _____¢ for ice cream, and had _____¢ left.

---

2. Two-ninths of the pupils of Edith's class were absent, and one-fourth of the pupils were at music practice. The class had less than 40 pupils.

What can you say?

If there were 27 pupils then _____ pupils would be absent. But there couldn't be 27 pupils, because "one-fourth of 27 pupils" wouldn't make much sense.

If there were 8 pupils at music practice then there must be _____ pupils in the class. But "two-ninths" of that number of pupils doesn't make much sense.

There must be an _____ (odd or even) number of pupils absent.

If you think it through, you can be sure there are _____ pupils absent and _____ pupils at music practice . . . or a total of ____ pupils

missing from the class and _____ pupils in the room.

---

3. One-sixth of the days this month are Sundays and one-sixth are Saturdays. The rest are school days.

What can you say?

There are _____ days in each of 7 months, _____ days in each of 4 months, and _____ days in February (except in Leap Years, when there are _____ days in February).

So, we can be sure the month referred to is either _____, _____, _____, or _____.

There are _____ Sundays, _____ Saturdays, and _____ school days in the month.

We can be sure that the first day of the month was _____.

---

4. One box was ¾ full of red marbles. Another box of the same size was ⅔ full of blue marbles. The owner of the store filled the rest of both boxes with white marbles. He needed 7 white marbles.

What can you say?

The number of marbles in a full box must be divisible by both _____ and _____. Numbers less than 100 that can be divided evenly by both these numbers are ____, ____, ____, ____, ____, ____, ____, and ____.

If there were 24 in each box then there would be _____ red and _____ blue marbles. The owner would need _____ white marbles to fill both boxes.

There must be _____ in each full box. The owner needed _____ white ones to fill the box with the red marbles. He needed _____ white ones to fill the box with the blue marbles.

70

IN → 12 ft.
OUT → 14 ft.

$ 20 / $ 30

36 in. / 9 in.
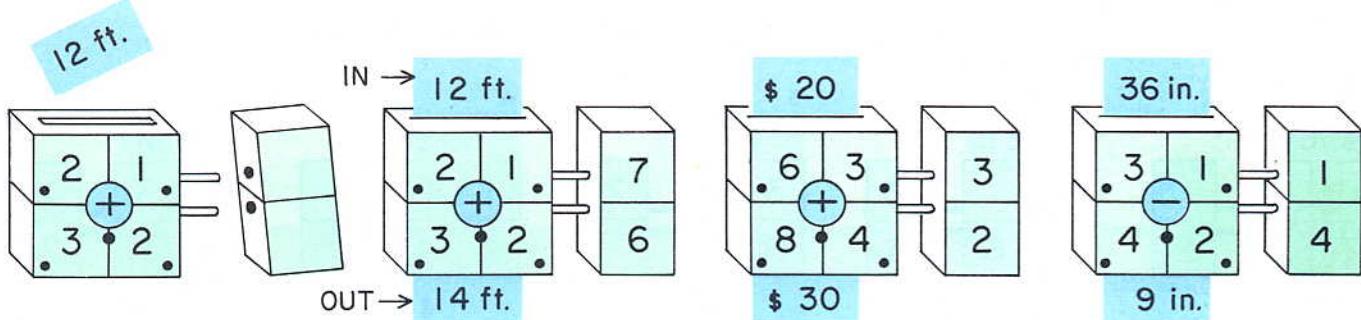
Mr. Wilson finds a SIMPLIFIER for Alec's FRACTION MACHINE.

30 oz. / 20 oz.

20 qt. / 12 qt.

42 ¢ / 7 ¢
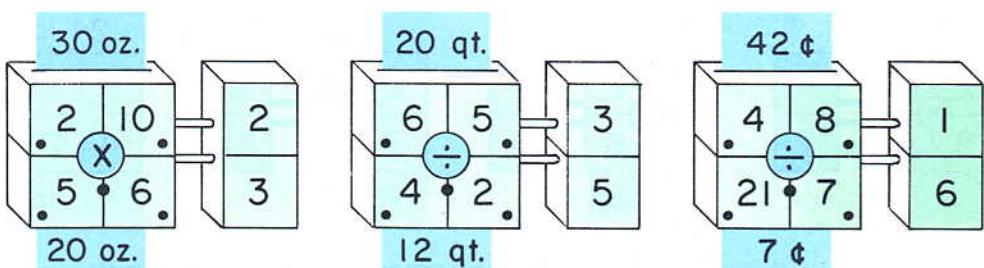
Have you any hunches about the Simplifier? What is its purpose?

"I thought those two pins on the side of Alec's machine were handles," Mr. Wilson admitted. "Then I found the 'Simplifier.'"

"As soon as I plugged the Simplifier onto those two pins, numbers appeared in its two windows. Examples of results I found are given above.

"I began to see what Alec had in mind when I kept notes on just the inputs, outputs, and the Simplifier.

"As I see it, the Simplifier shows a shortcut from the input to the output (as if it were very much like the simple fraction machine I experimented with first).

"Here are more of my records of experiments." *(Please fill in the missing entries.)*

9 doz.
3 4 | 5
(+)
9 3 | 3
doz.◄----

24 gal.
5 3 |
(+)
12 | 8
21 gal.◄----

12 ft.
(X) 7
(÷) 6
14 ft. ◄

$ 20
(X) 3
(÷) 2
$ 30 ◄

36 in.
(X) 1
(÷) 4
9 in. ◄

18 ft.
2 1 | 2
(−)
6 9 |
ft. ◄

16 oz.
3 1 |
(−)
8 |
4 oz. ◄

30 oz.
(X) 2
(÷) 3
oz.

20 qt.
3
5
qt.

42 ¢
1
6
¢

36 in.
6 8 | 4
(X)
4 | 3
in. ◄

42 ¢
6 3 |
(÷)
7 2 |
¢ ◄

71
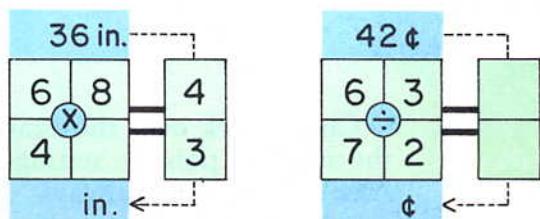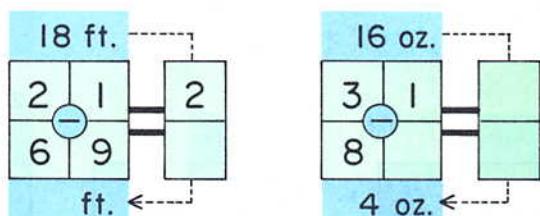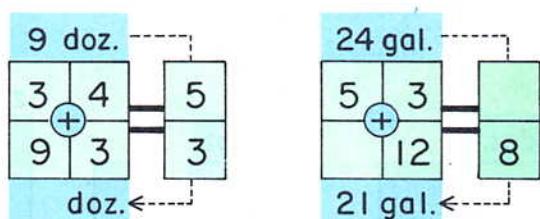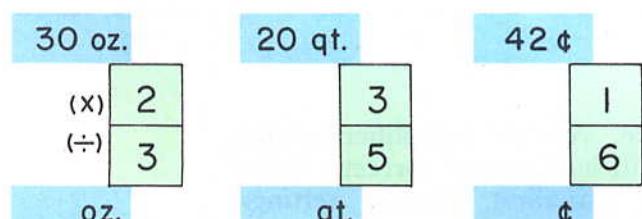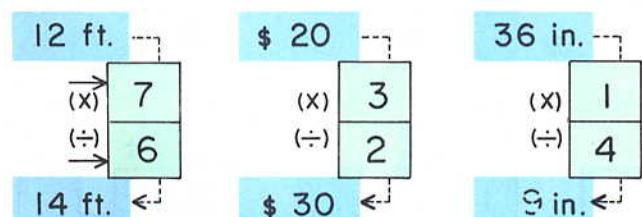
Use every shortcut you can to find the missing entries. Be sure to check your work.

1.
$ 6

| 1 | (+) | | | |
| 3 | 6 | 2 |

$ 3

2.
12 ft.

| 2 | 1 | | 2 |
| (+) | 6 | | 3 |

ft.

3.
16 oz.

| 3 | 1 | | 7 |
| (+) 8 | 2 | |

oz.

4.
60 ¢

| 1 | 1 | |
| (−) 2 | 4 |

¢

5.
qt.

| 7 | 3 | | 1 |
| (−) 8 | | 2 |

12 qt.

6.
40 in.

| 2 | 10 | |
| (×) 5 | 4 | 1 |

in.

7.
30 doz.

| 1 | 2 | |
| (×) | 6 |

5 doz.

8.
gal.

| 2 | 4 | | 8 |
| (×) 3 | | 21 |

16 gal.

9.
24 lb.

| 9 | 6 | |
| (÷) 4 | 2 |

lb.

10.
14 yd.

| 1 | 1 | |
| (÷) 2 | 6 |

yd.

Use any amounts you wish as inputs.

11. oz.

| 1 | 3 | |
| (+) 2 | 4 |

oz.

12. ¢

| 2 | 4 | |
| (+) 5 | 5 |

¢

13. ft.

| 3 | 3 | |
| (+) 2 | 2 |

ft.

14. $

| 1 | 1 | |
| (+) 3 | 4 |

$

15. in.

| 3 | 1 | |
| (+) 4 | 5 |

in.

16. ¢

| 3 | 1 | |
| (−) 4 | 2 |

¢

17. ft.

| 5 | 2 | |
| (−) 7 | 7 |

ft.

18. lb.

| 1 | 1 | |
| (−) 2 | 6 |

lb.

19. $

| 3 | 4 | |
| (−) 1 | 9 |

$

20. oz.

| 4 | 5 | |
| (−) 3 | 6 |

oz.

21. doz.

| 1 | 1 | |
| (×) 2 | 2 |

doz.

22. oz.

| 2 | 1 | |
| (×) 3 | 4 |

oz.

23. ¢

| 3 | 4 | |
| (×) 1 | 5 |

¢

24. ft.

| 5 | 4 | |
| (×) 6 | 3 |

ft.

25. $

| 2 | 9 | |
| (×) 3 | 8 |

$

26. $

| 9 | 3 | |
| (÷) 10 | 2 |

$

27. ft.

| 1 | 7 | |
| (÷) 2 | 4 |

ft.

28. hr.

| 2 | 7 | |
| (÷) 1 | 4 |

hr.

29. oz.

| 1 | 1 | |
| (÷) 2 | 5 |

oz.

30. ¢

| 7 | 2 | |
| (÷) 8 | 3 |

¢

Check back over the examples above. Are the Simplifier settings the smallest possible settings that would lead to the correct results? Alec's Simplifier always chooses the smallest possible settings.

"One day, during my experiments, it dawned on me that as soon as the 5 selector knobs on Alec's machine were set, the Simplifier was automatically set before I had selected an input. Also, as I changed inputs without changing settings, the Simplifier remained unchanged."

$$\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

24¢	12 hrs.	$6	60 min.
¢	hrs.	$	min.
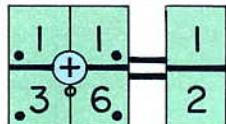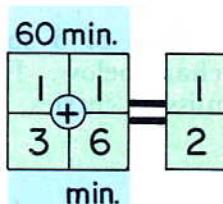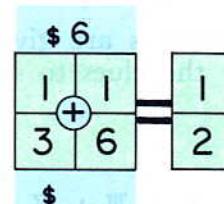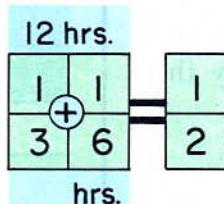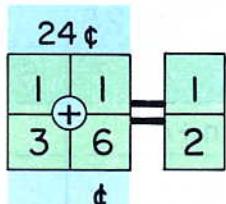
"Think of anything! Is $\frac{1}{3}$ of it plus $\frac{1}{6}$ of it equal to $\frac{1}{2}$ of it?"

A third of Bill's marbles are red. A sixth of them are blue. The rest are brown. What can you say?

If he has any marbles at all, the fewest he can have is _____ marbles. In that event, he would have _____ red and _____ blue marble(s).

If he had more than 20 but less than 30 marbles then he would have to have _____ red marbles and _____ blue marbles.

Can you be sure that one-half of the marbles are brown? _____ (Yes or no)

Can you say that Bill has twice as many red marbles as blue marbles? _____ (Yes or no)

If Bill has 7 blue marbles then he has a total of _____ marbles.

Bill put his red and brown marbles together. There were 30. Bill had _____ blue marbles.
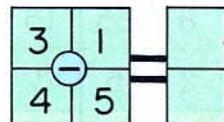
Mary and Jane collected stamps. They had the same number of stamps. Three-fourths of Mary's stamps were U. S. stamps. One-fifth of Jane's stamps were U. S. stamps. What can you say?

The fewest stamps each could have had is _____ stamps.

If that were the case, Mary had _____ U. S. stamps and Jane had _____ U. S. stamps. Mary had _____ more U. S. stamps than Jane had.
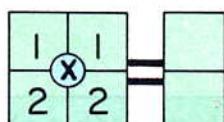
A half of a half-dollar is a _____.

Allen put half of his allowance in the bank. He spent half of the rest for a radio tube.

What can you say? _____

_____

Is it true that, for all numbers *a*, *b*, and *c* such that $c \neq 0$ and both *a* and *b* are multiples of *c*,

$$(a \div c) + (b \div c) = (a + b) \div c \; ?$$

What does the question mean?

Lists of 3 such numbers are given in the chart below. Follow the clues to supply the missing entries.

	I	II	III	IV	V	VI
a	6	20	15	60	63	40
b	8	35	15	40	28	56
c	2	5	3	10	7	8
a ÷ c	3	4				
b ÷ c	4	7				
(a÷c)+(b÷c)	7					
a + b	14	55				
(a + b) ÷ c	7					

In each column of the chart, is it true that

$$(a \div c) + (b \div c) = (a + b) \div c \; ?$$

Notice that each group of three numbers (*a*, *b*, and *c*) was selected so that both *a* and *b* could be evenly divided by *c*. This is because, at this point, we wish to avoid fractions. Why did we say that *c* should not be 0?

Here's another way to arrange our investigation:

a	b	c	(a÷c)+(b÷c)	(a+b)÷c
8	12	4	2 + 3	20 ÷ 4
15	21	3	5 +	÷
28	14	7	+	÷
18	12	3	+	÷
45	36	9	+	÷
72	24	8	+	÷

Make your own selections of three numbers (*a*, *b*, and *c*). Remember to choose them to avoid fractions and to avoid dividing by 0.

	VII	VIII	IX	X	XI	XII
a	42	30				
b	28					
c	7	5	3	9		
a ÷ c						
b ÷ c						
(a÷c)+(b÷c)						
a + b						
(a + b) ÷ c						

What is your conclusion?

Is it true that, for all *a*, *b*, and *c* such that $c \neq 0$, *a* and *b* are multiples of *c*, and $a > b$,

$$(a \div c) - (b \div c) = (a - b) \div c \; ?$$

Here is an incomplete record of an experiment:

a	b	c	(a÷c)−(b÷c)	(a−b)÷c
30	12	3	10 − 4	18 ÷ 3
40	15	5	−	÷
		4	15 − 7	÷

An incomplete record of another experiment:

a	42	63	56	
b	24	36		
c	6	9	7	2
a ÷ c	7			
b ÷ c	4			
(a÷c)−(b÷c)	3			
a − b	18			
(a−b)÷c				

What is your conclusion?

More experiments may convince you that, for all whole numbers a, b, and c such that a and b are multiples of c, $c \neq 0$, and $a \geq b$,

$$(a \div c) + (b \div c) = (a + b) \div c$$

and

$$(a \div c) - (b \div c) = (a - b) \div c,$$

Some mathematicians refer to these as patterns for the distributive principles for division over addition and over subtraction.

You have already used these principles many times — when you had such problems as $42 \div 3 = ?$ You might have thought about it this way:

$$3\overline{)42} \;=\; 3\overline{)30 + 12}^{\,10 + 4 \,=\, 14}$$

You could have recorded your computations in this way:

$$(30 \div 3) + (12 \div 3) = (30 + 12) \div 3$$

In arithmetic, when we write one number above and another below a bar, we agree that we mean to divide the top number (numerator) by the bottom number (denominator). We could write $12 \div 3$ in this way:

$$\frac{12}{3} \quad \begin{array}{l}\leftarrow \text{Numerator} \\ \leftarrow \text{Denominator}\end{array}$$

Alec would use other names:

$$\frac{12}{3} \quad \begin{array}{l}\leftarrow \text{Multiplier (machine)} \\ \leftarrow \text{Divider (machine)}\end{array}$$

So we could write :

$$\frac{30}{3} + \frac{12}{3} = \frac{30 + 12}{3} = \frac{42}{3}$$

and we could rewrite the patterns for the distributive principles:

$$(a \div c) + (b \div c) = (a + b) \div c$$
$$(a \div c) - (b \div c) = (a - b) \div c$$

in this way:

I. $$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$$

II. $$\frac{a}{c} - \frac{b}{c} = \frac{a - b}{c}$$

Let's experiment further.

			I $\dfrac{a}{c} + \dfrac{b}{c} = \dfrac{a+b}{c}$	II $\dfrac{a}{c} - \dfrac{b}{c} = \dfrac{a-b}{c}$
1.	a	30	$\dfrac{30}{3} + \dfrac{12}{3} = \dfrac{42}{3}$	$\dfrac{30}{3} - \dfrac{12}{3} = \dfrac{}{}$
	b	12		
	c	3	$10 + 4 =$	$\;-\; =$
2.	a	40	$\dfrac{40}{} + \dfrac{}{} = \dfrac{}{}$	$\dfrac{}{} - \dfrac{30}{} = \dfrac{}{}$
	b	30		
	c	5	$\;+\; =$	$\;-\; =$
3.	a	63	$\dfrac{}{} + \dfrac{}{} = \dfrac{}{}$	$\dfrac{}{} - \dfrac{}{} = \dfrac{}{}$
	b	21		
	c	7	$\;+\; =$	$\;-\; =$
4.	a	88	$\dfrac{}{} + \dfrac{}{} = \dfrac{}{}$	$\dfrac{}{} - \dfrac{}{} = \dfrac{}{}$
	b	56		
	c	8	$\;+\; =$	$\;-\; =$

More experiments.

	I	II
5.	$\dfrac{12}{2} + \dfrac{4}{2} = \dfrac{16}{2} = 8$	$\dfrac{12}{2} - \dfrac{4}{2} = \dfrac{8}{2} = 4$
6.	$\dfrac{45}{5} + \dfrac{15}{5} = \dfrac{}{5} =$	$\dfrac{45}{5} - \dfrac{15}{5} = \dfrac{}{} =$
7.	$\dfrac{42}{6} + \dfrac{18}{6} = \dfrac{}{} =$	$\dfrac{42}{6} - \dfrac{18}{6} = \dfrac{}{} =$
8.	$\dfrac{14}{4} + \dfrac{6}{4} = \dfrac{}{} =$	$\dfrac{14}{4} - \dfrac{6}{4} = \dfrac{}{} =$
9.	$\dfrac{33}{6} + \dfrac{15}{6} = \dfrac{}{} =$	$\dfrac{33}{6} - \dfrac{15}{6} = \dfrac{}{} =$
10.	$\dfrac{17}{2} + \dfrac{7}{2} = \dfrac{}{} =$	$\dfrac{17}{2} - \dfrac{7}{2} = \dfrac{}{} =$

Do you notice a difference between examples above the heavy line and below it?

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \quad \text{and} \quad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

Let's check the results in examples similar to the last ones on the previous page.

$$\frac{6}{4} + \frac{2}{4} = \frac{}{4} = \qquad \left| \qquad \frac{6}{4} - \frac{2}{4} = \frac{}{4} = \right.$$

Suppose we are talking about gallons of milk. We know that a fourth or quarter of a gallon is called a "quart."

Mrs. Jones had 6 quarts of milk and bought 2 more quarts of milk. Together she had _____ quarts of milk, which is the same as _____ gallons.

Mrs. Brown had 6 quarts of milk and used 2 of them. She has _____ quarts left, which is the same as _____ gallon.

$$\frac{9}{2} + \frac{3}{2} = \frac{}{2} = \qquad \left| \qquad \frac{9}{2} - \frac{3}{2} = \frac{}{2} = \right.$$

Marge had 9 half-dollars and received 3 half-dollars for her birthday. That is a total of _____ half-dollars, which is the same as $_____.

A nickel is worth half as much as a dime. Al had 9 nickels and spent 3 of them; so he had _____ nickels left. He changed them into _____ dimes.

$$\frac{8}{10} + \frac{7}{10} = \frac{}{10} = 1\frac{1}{2} \quad \left| \quad \frac{8}{10} - \frac{7}{10} = \frac{}{10} \right.$$

Harry rode 8-tenths of a mile and walked 7-tenths of a mile further. That's a total of _____-tenths of a mile, which is a mile and a half.

It is 8-tenths of a mile to the library. After Sue had walked 7-tenths of a mile, she had _____-tenth left to walk.

Thus far, we have been adding and subtracting pairs of fractions in which the denominators are the same.)

$$\text{Numerator} \rightarrow \frac{7}{4} + \frac{9}{4} \begin{pmatrix} \leftarrow \text{Multiplier} \\ \leftarrow \text{Divider} \end{pmatrix}$$
$$\text{Denominator} \rightarrow$$

(Of course, Alec would say that the "dividers" are the same.)

Let's consider examples of addition and subtraction of fractions in which the denominators are not alike, such as:

$$\frac{1}{3} + \frac{5}{4} = \qquad \frac{7}{5} - \frac{2}{3} = \qquad \frac{3}{8} + \frac{5}{6} = \qquad \frac{4}{7} - \frac{1}{4} =$$

Many fractions are equivalent to $\frac{1}{3}$. Alec would say that many pairs of machine-settings would produce the same results as $\frac{1}{3}$.

The same can be said of all fractions such as $\frac{5}{4}, \frac{7}{3}, \frac{2}{3}$, etc.

$$\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{} = \frac{5}{} = \frac{6}{} = \frac{7}{} \text{ etc.}$$

$$\frac{5}{4} = \frac{10}{8} = \frac{15}{} = \frac{20}{20} = \frac{}{24} = \frac{}{28} \text{ etc.}$$

To find the sum of $\frac{1}{3}$ and $\frac{5}{4}$, we scan both lists of fractions until we find one in the list for $\frac{1}{3}$ that has the same denominator as one in the list for $\frac{5}{4}$.

$$\frac{1}{3} + \frac{5}{4}$$

$$\frac{4}{12} + \frac{}{12} =$$

The problem has been changed to a familiar form:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$\frac{7}{5} - \frac{2}{3} = \ ?$$

a. | $\frac{7}{5}$ | $\frac{14}{10}$ | $\frac{21}{?}$ | $\frac{28}{?}$ | $\frac{35}{?}$ | | | | etc.

(row a): $\frac{7}{5}$, $\frac{14}{10}$, $\frac{21}{\ }$, $\frac{28}{\ }$, $\frac{35}{\ }$, $\frac{\ }{30}$, $\frac{\ }{35}$, $\frac{\ }{40}$ etc.

b. $\frac{2}{3}$, $\frac{4}{6}$, $\frac{\ }{9}$, — — — — — etc.

c. $\frac{7}{5} - \frac{2}{3} = \underline{\ \ } - \underline{\ \ } = \underline{\ \ }$

The lists below can be used to help solve the problems that follow.

d. $\frac{3}{8}$, $\frac{6}{16}$, $\frac{9}{\ }$, — — — — — —

e. $\frac{5}{6}$, $\frac{10}{12}$, $\frac{15}{\ }$, — — — — — —

f. $\frac{4}{7}$, $\frac{\ }{14}$, — — — — — —

g. $\frac{1}{4}$, $\frac{\ }{8}$, — — — — — —

h. $\frac{7}{12}$, $\frac{\ }{24}$, — — — — — —

i. $\frac{8}{9}$, $\frac{\ }{18}$, — — — — — —

j. $\frac{3}{8} + \frac{5}{6} = \underline{\ \ } + \underline{\ \ } = \underline{\ \ }$

k. $\frac{4}{7} - \frac{1}{4} = \underline{\ \ } - \underline{\ \ } = \underline{\ \ }$

l. $\frac{8}{9} + \frac{7}{12} = \underline{\ \ } + \underline{\ \ } = \underline{\ \ }$

m. $\frac{5}{6} - \frac{4}{7} = \underline{\ \ } - \underline{\ \ } = \underline{\ \ }$

Whole numbers can be written as fractions.

$$3 = 3 \div 1 = \frac{3}{1} \text{ also } 3 = 6 \div 2 = \frac{6}{2} \text{ etc.}$$

n. $3 = \frac{3}{1} = \frac{6}{2} = \frac{9}{\ } = \frac{\ }{4} = \frac{\ }{5} = $ etc.

o. 1 | $\frac{1}{1}$ | $\frac{2}{2}$ | $\frac{\ }{3}$ | $\frac{\ }{4}$ | $\frac{\ }{5}$ | $\frac{\ }{6}$ | $\frac{\ }{7}$

p. 8 | $\frac{\ }{1}$ | $\frac{\ }{2}$ | $\frac{\ }{3}$ | $\frac{\ }{4}$ | $\frac{\ }{5}$ | $\frac{\ }{6}$ | $\frac{\ }{7}$

We use a shorthand that combines a whole number and a fraction:

$3\frac{1}{2}$ means $3 + \frac{1}{2}$ $8\frac{1}{3}$ means $8 + \frac{1}{3}$

q. $1\frac{4}{7}$ means $\underline{\ \ } + \underline{\ \ }$ r. $3\frac{5}{6}$ means $\underline{\ \ } + \underline{\ \ }$

We can rewrite such shorthand expressions as single fractions in this way:

s. $3\frac{3}{8} = 3 + \frac{3}{8} = \frac{3}{1} + \frac{3}{8} = \frac{24}{8} + \frac{\ }{8} = \frac{\ }{8}$

t. $8\frac{1}{2} = 8 + \frac{1}{2} = \frac{\ }{1} + \frac{\ }{2} = \underline{\ \ } + \underline{\ \ } = \underline{\ \ }$

u. $1\frac{2}{3} = 1 + \frac{2}{3} = \frac{\ }{1} + \frac{\ }{3} = \underline{\ \ } + \underline{\ \ } = \underline{\ \ }$

v. $8\frac{1}{2}$ | $\frac{17}{2}$ | $\frac{\ }{4}$ | $\frac{\ }{6}$ | $\frac{\ }{8}$ | — — —

w. $1\frac{2}{3}$ | $\frac{\ }{3}$ | $\frac{10}{6}$ | $\frac{\ }{9}$ | $\frac{\ }{12}$ | — — —

x. $8\frac{1}{2} + 1\frac{2}{3} = \underline{\ \ } + \underline{\ \ } = \underline{\ \ }$

y. $8\frac{1}{2} - 1\frac{2}{3} = \underline{\ \ } - \underline{\ \ } = \underline{\ \ }$

There is an alternate approach that often saves much work. (See next page.)

Please supply the missing entries.

A.

$8\frac{1}{5} \rightarrow 8 + \frac{1}{5}$

$+2\frac{3}{5} \rightarrow 2 + \frac{3}{5}$

$10\frac{4}{5} \leftarrow 10 + \frac{4}{5}$

B.

$3\frac{2}{9} \rightarrow 3 + —$

$+1\frac{5}{9} \rightarrow \quad + —$

$— \leftarrow \quad + —$

C.

$4\frac{2}{3} \rightarrow 4 + —$

$-1\frac{1}{3} \rightarrow \quad + —$

$— \leftarrow \quad + —$

D.

$13\frac{6}{7} \rightarrow 13 + —$

$-9\frac{2}{7} \rightarrow \quad + —$

$— \leftarrow \quad + —$

Sometimes, an additional step will reduce the results to a simpler form.

E.

$5\frac{1}{9} \longrightarrow \quad + —$

$+3\frac{2}{9} \longrightarrow \quad + —$

$— \leftarrow 8 + \frac{1}{3} \leftarrow \quad + \frac{}{9}$

F.

$2\frac{4}{5} \longrightarrow 2 + —$

$+1\frac{3}{5} \longrightarrow 1 + —$

$— \leftarrow \quad + — \leftarrow \quad + —$

G.

$1\frac{3}{4} \longrightarrow 1 + —$

$+6\frac{3}{4} \longrightarrow 6 + —$

$— \leftarrow \quad + — \leftarrow \quad + —$

H.

$5\frac{3}{4} \longrightarrow 5 + \frac{}{4}$

$-2\frac{1}{4} \longrightarrow 2 + \frac{}{4}$

$— \leftarrow 3 + \frac{1}{2} \leftarrow 3 + \frac{}{4}$

I.

$3\frac{7}{8} \longrightarrow \quad + —$

$-1\frac{1}{8} \longrightarrow \quad + —$

$2 — \leftarrow 2 + — \leftarrow 2 + —$

J.

$2\frac{5}{9} \longrightarrow \quad + —$

$+3\frac{7}{9} \longrightarrow \quad + —$

$— \leftarrow \quad \leftarrow 5 + —$

Sometimes, a new problem arises in a subtraction example.

K.

$5\frac{1}{4} \rightarrow 4 + 1 + \frac{1}{4} \rightarrow 4 + \frac{5}{4}$

$-1\frac{3}{4} \longrightarrow 1 + \frac{}{4}$

$— \leftarrow 3 + \frac{}{2} \leftarrow 3 + \frac{}{4}$

L.

$7\frac{3}{8} \longrightarrow 6 + 1 + \frac{3}{8} \longrightarrow 6 + \frac{}{8}$

$-6\frac{5}{8} \longrightarrow \quad + \frac{}{8}$

$— \leftarrow \quad + — \leftarrow \quad + \frac{}{8}$

Use scratch paper if necessary to find the results.

M.

$3\frac{1}{7}$
$+2\frac{4}{7}$

N.

$9\frac{1}{4}$
$+5\frac{1}{4}$

O.

$8\frac{3}{5}$
$+9\frac{4}{5}$

P.

$5\frac{1}{6}$
$+3\frac{5}{6}$

Q.

$197\frac{2}{3}$
$+18\frac{2}{3}$

R.

$3\frac{8}{9}$
$+1\frac{7}{9}$

S.

$15\frac{5}{8}$
$+28\frac{7}{8}$

T.

$10\frac{5}{9}$
$-3\frac{1}{9}$

U.

$38\frac{3}{4}$
$-8\frac{1}{4}$

V.

$4\frac{5}{6}$
$-1\frac{1}{6}$

W.

$8\frac{1}{3}$
$-5\frac{2}{3}$

X.

$21\frac{1}{8}$
$-7\frac{3}{8}$

Y.

$14\frac{2}{5}$
$-3\frac{4}{5}$

Z.

$18\frac{1}{6}$
$-9\frac{5}{6}$

78

Denominators of fractions may be different:

A.
$$1\frac{1}{2} \rightarrow 1\frac{2}{4}$$
$$+4\frac{1}{4} \rightarrow 4\frac{1}{4}$$
$$5\frac{3}{4} \leftarrow 5\frac{3}{4}$$

B.
$$4\frac{1}{6} \rightarrow 4\frac{\ }{\ }$$
$$+2\frac{1}{3} \rightarrow \ \frac{\ }{\ }$$
$$\frac{\ }{\ } \leftarrow \frac{\ }{\ }$$

C.
$$7\frac{1}{2} \rightarrow 7\frac{\ }{6}$$
$$+1\frac{1}{3} \rightarrow \ \frac{\ }{6}$$
$$\frac{\ }{6} \leftarrow \frac{\ }{6}$$

D.
$$5\frac{3}{4} \rightarrow \ \frac{\ }{\ }$$
$$+5\frac{3}{8} \rightarrow \ \frac{\ }{\ }$$
$$\frac{\ }{\ } \leftarrow \frac{\ }{\ }$$

E.
$$7\frac{3}{8} \rightarrow 7\frac{3}{8}$$
$$-3\frac{1}{4} \rightarrow 3\frac{2}{8}$$
$$\frac{\ }{\ } \leftarrow 4\frac{\ }{8}$$

F.
$$8\frac{2}{3} \rightarrow 8\frac{\ }{\ }$$
$$-5\frac{1}{2} \rightarrow \ \frac{\ }{\ }$$
$$\frac{\ }{\ } \leftarrow \frac{\ }{\ }$$

G.
$$4\frac{5}{6} \rightarrow 4\frac{\ }{\ }$$
$$-1\frac{1}{3} \rightarrow \ \frac{\ }{\ }$$
$$\frac{\ }{\ } \leftarrow \frac{\ }{\ }$$

H.
$$6\frac{3}{4} \rightarrow 6\frac{\ }{\ }$$
$$-2\frac{1}{3} \rightarrow \ \frac{\ }{\ }$$
$$\frac{\ }{\ } \leftarrow \frac{\ }{\ }$$

I.
$$9\frac{1}{2} \rightarrow 9\frac{2}{4} \rightarrow 8\frac{6}{4}$$
$$-2\frac{3}{4} \rightarrow 2\frac{3}{4} \rightarrow 2\frac{\ }{4}$$
$$\frac{\ }{\ } \leftarrow \ \frac{\ }{\ } \ 6\frac{\ }{4}$$

J.
$$8\frac{1}{3} \rightarrow 8\frac{\ }{6} \rightarrow 7\frac{\ }{\ }$$
$$-1\frac{5}{6} \rightarrow 1\frac{\ }{\ } \rightarrow 1\frac{\ }{\ }$$
$$\frac{\ }{\ } \leftarrow \ \frac{\ }{\ } \ 5\frac{\ }{\ }$$

K.
$$6\frac{1}{3} \rightarrow 6\frac{\ }{\ } \rightarrow 5\frac{\ }{\ }$$
$$-2\frac{3}{4} \rightarrow 2\frac{\ }{\ } \rightarrow \ \frac{\ }{\ }$$
$$\frac{\ }{\ } \leftarrow \frac{\ }{\ }$$

L.
$$10\frac{3}{8} \rightarrow \ \frac{\ }{24} \rightarrow \ \frac{\ }{\ }$$
$$-9\frac{2}{3} \rightarrow \ \frac{\ }{\ } \rightarrow \ \frac{\ }{\ }$$
$$\frac{\ }{\ } \leftarrow \frac{\ }{\ }$$

M.
$$5\frac{1}{5} \rightarrow \ \frac{\ }{\ } \rightarrow \ \frac{\ }{\ }$$
$$-1\frac{4}{7} \rightarrow \ \frac{\ }{\ } \rightarrow \ \frac{\ }{\ }$$
$$\frac{\ }{\ } \leftarrow \frac{\ }{\ }$$

N.
$$15\frac{1}{6} \rightarrow \ \frac{\ }{\ } \rightarrow \ \frac{\ }{\ }$$
$$-7\frac{7}{9} \rightarrow \ \frac{\ }{\ } \rightarrow \ \frac{\ }{\ }$$
$$\frac{\ }{\ } \leftarrow \frac{\ }{\ }$$

You may wish to use the blank spaces below for scratch work to get your results.

O.
$$7\frac{2}{3}$$
$$+2\frac{3}{4}$$

P.
$$15\frac{3}{10}$$
$$-7\frac{3}{4}$$

Q.
$$12\frac{1}{6}$$
$$-5\frac{4}{9}$$

R.
$$9\frac{1}{3}$$
$$+9\frac{2}{5}$$

S.
$$13\frac{7}{8}$$
$$+7\frac{5}{6}$$

T.
$$8\frac{3}{4}$$
$$-\ \frac{6}{7}$$

$\frac{1}{4} + \frac{1}{2} =$

(1)	(2)	(3)	(4)
$\frac{2}{6}$	$\frac{3}{4}$	$\frac{1}{8}$	$\frac{2}{2}$

Mark correct answer → ↓   ↓ ← Cross out furthest from correct answer

(1) (2) (3) (4) $\frac{5}{8}$ ← Difference between the answers you marked
○ ● ⊗ ○

1 2
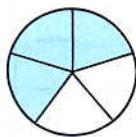
What part of the sketch is shaded?

(1)	(2)	(3)	(4)
$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{3}{2}$

(1) (2) (3) (4)
○ ○ ○ ○ _ _ _ _ _

$\frac{5}{6} - \frac{1}{3} =$

(1)	(2)	(3)	(4)
$\frac{4}{3}$	$\frac{6}{9}$	$\frac{5}{18}$	$\frac{1}{2}$

(1) (2) (3) (4)
○ ○ ○ ○ _ _ _ _ _

3

Which fraction is more than $1\frac{1}{8}$, but less than $1\frac{1}{4}$?

(1)	(2)	(3)	(4)
$1\frac{1}{6}$	$1\frac{1}{2}$	$1\frac{1}{3}$	$1\frac{1}{10}$

(1) (2) (3) (4)
○ ○ ○ ○ _ _ _ _ _

4 5

Which is the smallest fraction?

(1)	(2)	(3)	(4)
$\frac{6}{9}$	$\frac{8}{15}$	$\frac{9}{12}$	$\frac{6}{4}$

(1) (2) (3) (4)
○ ○ ○ ○ _ _ _ _

Which fraction is equal to $3\frac{3}{5}$?

(1)	(2)	(3)	(4)
$\frac{11}{5}$	$\frac{6}{5}$	$\frac{33}{5}$	$\frac{18}{5}$

(1) (2) (3) (4)
○ ○ ○ ○ _ _ _ _

6

Which fraction contains 3 as a common factor of both numerator and denominator?

(1)	(2)	(3)	(4)
$\frac{12}{18}$	$\frac{9}{13}$	$\frac{5}{15}$	$\frac{10}{24}$

(1) (2) (3) (4)
○ ○ ○ ○ _ _ _ _ _

7 8

Which figure is $\frac{1}{4}$ of the tinted figure?

A   B

C   D

(1) A (2) B (3) C (4) D

(1) (2) (3) (4)
○ ○ ○ ○ _ _ _ _

Which addition example is represented by the diagram?

(1) $\frac{1}{2} + \frac{1}{4}$ (3) $\frac{1}{3} + \frac{1}{4}$

(2) $\frac{4}{5} + \frac{3}{6}$ (4) $\frac{4}{9} + \frac{1}{3}$

(1) (2) (3) (4)
○ ○ ○ ○ _ _ _ _

9

How many hours are there between 10:30 A.M. and 4:15 P.M. on the same day?

(1) $5\frac{3}{4}$ (3) $14\frac{3}{4}$

(2) $6\frac{1}{4}$ (4) $4\frac{3}{4}$

(1) (2) (3) (4)
○ ○ ○ ○ _ _ _ _

10 11

Bernie had a launching tower for a model rocket that was $18\frac{1}{4}$ inches tall. Carl, his friend, had one that was $32\frac{3}{4}$ inches tall. How many inches taller is Carl's tower?

(1) $14\frac{1}{2}$ (2) 51   (3) $18\frac{1}{2}$ (4) 14

(1) (2) (3) (4)
○ ○ ○ ○ _ _ _ _ _

Mrs. Burns had a piece of cloth $3\frac{1}{8}$ yards long. She used $1\frac{1}{4}$ yards to make a blouse. How many yards of material did she have left over?

(1) $4\frac{3}{8}$ (2) $2\frac{1}{8}$ (3) $1\frac{7}{8}$ (4) $2\frac{1}{12}$

(1) (2) (3) (4)
○ ○ ○ ○ _ _ _ _ _ _

12 13

A recipe designed to serve 6 calls for $3\frac{1}{2}$ cups of water. Mrs. Barnes had 9 to serve, so she increased the $3\frac{1}{2}$ cups of water to a total of how many cups?

(1) $6\frac{1}{2}$ (2) $5\frac{1}{4}$ (3) 5   (4) $4\frac{1}{4}$

(1) (2) (3) (4)
○ ○ ○ ○ _ _ _ _ _

Bill's dog eats $1\frac{1}{4}$ cans of dog food a day. The price of the dog food was 3 cans for 50¢. How much does it cost him to feed his dog for 12 days?

(1) $ 2.00 (2) $ 12.25 (3) $ 2.50 (4) $ 7.50

(1) (2) (3) (4)
○ ○ ○ ○ $_ _ _ _ _ _

I'll refer to each of the following as a "system."

I'll experiment to find out what these systems do when I consider a number of things or a number of units of measure, such as

4 , 1 , 17 , 105 etc.

I'll record my experiments in this way:

1. IN $\boxed{x2}$ OUT 9 | 8   24 $\boxed{\div3}$ 8   6 $\boxed{\begin{array}{c}x3\\\div2\end{array}}$ 9

From here on, I'll omit the '×' and '÷' signs.

> Let's agree: Numbers written in the green block above the line are multipliers; those written below the line are dividers.

I'll not worry about the order I follow in using systems that have more than one operation.

2. 6 $\boxed{\begin{array}{c}3\\|8\\2\end{array}}$ 9   6 $\boxed{\begin{array}{c}2\\3\\3\end{array}}$ 9

6 x 3 = 18 ; 18 ÷ 2 = 9   6 ÷ 2 = 3 ; 3 x 3 = 9

Let's try a larger system: $\boxed{\begin{array}{cc}2&7\\3&4\end{array}}$

3. 12 $\boxed{\begin{array}{c}2\\3\end{array}}$ 24 8 $\boxed{\begin{array}{c}7\\4\end{array}}$ 56 | 4

4. 12 $\boxed{\begin{array}{c}3\\4\quad3\end{array}}$ | $\boxed{\begin{array}{c}7\\2\end{array}}$

5. 12 $\boxed{\begin{array}{c}7\\4\end{array}}$ $\boxed{\begin{array}{c}2\\3\end{array}}$

> A lot of experimenting leads me to believe that the order in which I follow the directions given in a system does not affect the outcome at all.

However, in a simple 2-operation system involving both division and multiplication, I'll usually do the division first (for two reasons):

(1) I'll want my 2-operation system to behave like what people call "fractions." By $\frac{2}{3}$ of 60 minutes, people mean: one-third of 60 minutes is 20 minutes (÷ 3) and $\frac{2}{3}$ is twice as much, or 40 minutes (× 2). So, in this example:

60 $\boxed{\begin{array}{c}2\\3\end{array}}$ 40

I'll divide by 3 and multiply the result by 2.

(2) It will frequently lead to simpler arithmetic.

Here are some 4-operation systems and an example of one use of each.

a. 6 $\boxed{\begin{array}{cc}3&2\\1&1\end{array}}$ 36   b. 6 $\boxed{\begin{array}{cc}3&1\\1&2\end{array}}$

c. 6 $\boxed{\begin{array}{cc}2&1\\3&1\end{array}}$ d. 6 $\boxed{\begin{array}{cc}1&1\\2&3\end{array}}$

e. 30 $\boxed{\begin{array}{cc}2&2\\3&5\end{array}}$ f. 50 $\boxed{\begin{array}{cc}3&2\\2&5\end{array}}$

g. 24 $\boxed{\begin{array}{cc}9&2\\8&3\end{array}}$ h. 12 $\boxed{\begin{array}{cc}5&8\\4&6\end{array}}$
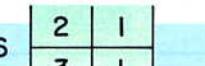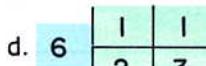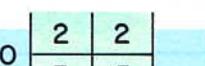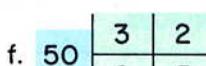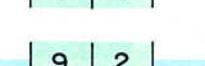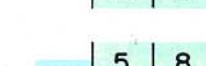
Notice that I can find 2-operation systems that do the same jobs as each of the above systems.

A. 6 $\boxed{\begin{array}{c}6\\1\end{array}}$ 36   B. 6 $\boxed{\begin{array}{c}\\2\end{array}}$ 9

C. 6 $\boxed{\begin{array}{c}\\3\end{array}}$ 4   D. 6 $\boxed{\begin{array}{c}|\\\end{array}}$ 1

E. 30 $\boxed{\begin{array}{c}\\15\end{array}}$ 8   F. 50 $\boxed{\quad}$ 60

G. 24 $\boxed{\quad}$ 18   H. 12 $\boxed{\quad}$ 20

> I have a hunch that I can find a 2-operation system for every 4-operation system so that both systems produce the same results.

Consider 2-operation systems in which both operations are either both multiplier machines or both divider machines.

If 2 systems always produce the same results, I call them "equal."

1. $7 \fbox{2|5} 70$, $7 \fbox{10}$, $\fbox{2|5} = \fbox{10}$

2. $5 \fbox{4|3} 60$, $5 \fbox{\ \ } 60$, $\fbox{4|3} = \fbox{\ }$

3. $24 \fbox{4|2} 3$, $24 \fbox{\ \ } 3$, $\fbox{4|2} = \fbox{\ }$

4. $45 \fbox{5|3}$, $45 \fbox{\ \ }$, $\fbox{5|3} = \fbox{\ }$

5. $24 \fbox{$\frac{1|5}{4|3}$} 10$, $24 \fbox{12} 10$, $\fbox{$\frac{1|5}{4|3}$} = \fbox{\ }$

6. $45 \fbox{$\frac{4|2}{5|3}$}$, $45 \fbox{\ \ }$, $\fbox{\ \ } = \fbox{\ }$

7. $18 \fbox{$\frac{5|4}{2|3}$}$, $18 \fbox{\ \ }$, $\fbox{\ \ } = \fbox{\ }$

8. $60 \fbox{$\frac{3|4}{5|6}$}$, $60 \fbox{\ \ }$, $\fbox{\ \ } = \fbox{\ }$

9. $60 \fbox{$\frac{2|1}{1|5}$}$, $60 \fbox{\ \ }$, $\fbox{\ \ } = \fbox{\ }$

Those last two examples convince me that

$$\fbox{$\frac{3|4}{5|6}$} = \fbox{$\frac{2|1}{1|5}$} = \fbox{$\frac{12}{30}$} = \fbox{$\frac{2}{5}$}$$

Clearly, the last of the 4 equal systems is simpler than the others. How do you find simpler systems?

It may help to notice this:

10. $7 \fbox{$\frac{1}{1}$} 7$ $27 \fbox{$\frac{9}{9}$}$ $42 \fbox{$\frac{13}{13}$}$

Here are some experiments:

11. $50 \fbox{$\frac{6}{10}$}$ $50 \fbox{$\frac{2|3}{2|5}$}$ $50 \fbox{$\frac{3}{5}$}$

12. $42 \fbox{$\frac{12}{42}$}$ $42 \fbox{$\frac{6}{6}$}$ $42 \fbox{\ \ }$

13. $60 \fbox{$\frac{16}{20}$}$ $60 \fbox{$\frac{\ }{5}$}$ $60 \fbox{\ \ }$

14. $72 \fbox{$\frac{6}{9}$}$ $72 \fbox{$\frac{3|\ }{\ |\ }$}$ $72 \fbox{\ \ }$

↑

If I use the largest common factor of both the divider and multiplier machines, I will find the simplest 2-operation system.

This same idea, together with the in-any-order rule, can help simplify 4-operation systems.

15. $18 \fbox{$\frac{5|3}{3|2}$}$, $18 \fbox{$\frac{3|5}{3|2}$}$, $18 \fbox{\ \ }$

16. $12 \fbox{$\frac{4|7}{1|4}$}$, $12 \fbox{\ \ \ }$, $12 \fbox{\ \ }$

From now on, I'll use this shorthand in systems like this:

17. $24 \fbox{$\frac{7|\cancel{6}^{1}}{\cancel{5}_{1}|6}$}$, $24 \fbox{$\frac{7}{6}$}$

I'll use a similar shorthand to keep track when taking out the largest common factor:

18. $24 \fbox{$\frac{\overset{3}{\cancel{9}}}{\underset{4}{\cancel{12}}}$}$ $24 \fbox{$\frac{3}{4}$}$

Sometimes I find more than one opportunity to use this idea in a single example.

19. $70 \fbox{$\frac{\overset{1}{\cancel{7}}|\overset{1}{\cancel{5}}}{\underset{2}{\cancel{14}}|\underset{1}{\cancel{5}}}$}$ $70 \fbox{$\frac{1}{\ }$}$

(continued next page)

Here are further experiments:

1.
$$35 \begin{array}{|c|c|} \hline \cancel{15}\,\cancel{7} & \cancel{7}\,\cancel{5} \\ \hline 7 & 5 \\ \hline \end{array} \quad , \quad 35 \begin{array}{|c|} \hline \\ \hline \end{array}$$

2.
$$30 \begin{array}{|c|c|} \hline 24 & 35 \\ \hline 25 & 18 \\ \hline \end{array} \quad , \quad 30 \begin{array}{|c|} \hline \\ \hline \end{array}$$

3.
$$30 \begin{array}{|c|c|} \hline \cancel{4}\,27 & 27 \\ \hline \cancel{15}\,\cancel{6} & \cancel{6} \\ \hline \end{array} \quad , \quad 30 \begin{array}{|c|c|} \hline 2 & \cancel{9} \\ \hline 5 & \cancel{3} \\ \hline \end{array} \quad , \quad 30 \begin{array}{|c|} \hline \\ \hline \end{array}$$

I will now omit the inputs and outputs (unless I am not sure that two systems are equal).

4.
$$\begin{array}{|c|c|} \hline 7 & 3 \\ \hline 10 & 7 \\ \hline \end{array} = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

5.
$$\begin{array}{|c|c|} \hline 8 & 10 \\ \hline 7 & 15 \\ \hline \end{array} = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

6.
$$\begin{array}{|c|c|} \hline 3 & 2 \\ \hline 7 & 5 \\ \hline \end{array} = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

7.
$$\begin{array}{|c|c|} \hline 3 & 4 \\ \hline 10 & 9 \\ \hline \end{array} = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

8.
$$\begin{array}{|c|c|} \hline 14 & 7 \\ \hline 21 & 35 \\ \hline \end{array} = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

9.
$$\begin{array}{|c|c|} \hline 27 & 25 \\ \hline 5 & 36 \\ \hline \end{array} = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

10.
$$\begin{array}{|c|c|} \hline 25 & 12 \\ \hline 24 & 7 \\ \hline \end{array} = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

11.
$$\begin{array}{|c|c|} \hline 3 & 4 \\ \hline 7 & 12 \\ \hline \end{array} = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

12.
$$\begin{array}{|c|c|} \hline 17 & 2 \\ \hline 51 & 5 \\ \hline \end{array} = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

13.
$$\begin{array}{|c|c|} \hline 26 & 10 \\ \hline 15 & 39 \\ \hline \end{array} = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

14.
$$\begin{array}{|c|c|} \hline 7 & 3 \\ \hline 8 & 2 \\ \hline \end{array} = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

15.
$$\begin{array}{|c|c|} \hline 8 & 27 \\ \hline 30 & 12 \\ \hline \end{array} = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

16.
$$32 \begin{array}{|c|c|} \hline 7 & 3 \\ \hline 8 & 2 \\ \hline \end{array} \quad , \quad 32 \begin{array}{|c|} \hline 21 \\ \hline 16 \\ \hline \end{array}$$

17.
$$90 \begin{array}{|c|c|} \hline 8 & 27 \\ \hline 30 & 12 \\ \hline \end{array} \quad , \quad 90 \begin{array}{|c|} \hline 3 \\ \hline 5 \\ \hline \end{array}$$

(Note: I could make a machine that could do this kind of arithmetic.)

When people speak about "$\frac{2}{3}$ of a dozen" (12 eggs), or "$\frac{1}{4}$ of a yard" (36 inches), or "$\frac{7}{8}$ of a day" (24 hours), they are thinking along the lines of my 2-operation system.

(They say) (They write) (My system)

18. $\frac{2}{3}$ of a dozen .... $12 \times \frac{2}{3}$ .... $12 \begin{array}{|c|} \hline \times 2 \\ \hline \div 3 \\ \hline \end{array} 8$

19. $\frac{1}{4}$ of a yard .... $36 \times \frac{1}{4}$ ...... $36 \begin{array}{|c|} \hline \times 1 \\ \hline \div 4 \\ \hline \end{array}$

20. $\frac{7}{8}$ of a day ....$24 \times \frac{7}{8}$ ...... $24 \begin{array}{|c|} \hline \times 7 \\ \hline \div 8 \\ \hline \end{array}$

People say: "The class had $\frac{2}{3}$ of an hour to do the test. Bill used up $\frac{4}{5}$ of the time allowed. How much time did Bill use?"

> In finding the answer, people say: "To find $\frac{2}{3}$ of 60 minutes, I first find $\frac{1}{3}$ of 60 — $60 \div 3 = 20$. Now $\frac{2}{3}$ must be $20 \times 2 = 40$. To find $\frac{4}{5}$ of 40 minutes, I will find $\frac{1}{5}$ first: $40 \div 5 = 8$; and $\frac{4}{5}$ will be 4 times as much: $8 \times 4 = 32$. So, Bill must have used _____ minutes."

They follow the same steps I do when I use my 4-operation system with "60 minutes" as the input:

$$60 \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 3 & 5 \\ \hline \end{array} \quad \text{or} \quad 60 \begin{array}{|c|} \hline 8 \\ \hline 15 \\ \hline \end{array}$$

When people speak about "Multiplication of Fractions," such as:

$$\frac{2}{3} \times \frac{4}{5} = \text{—}$$

they ask the same kind of questions I ask when I want to find simpler systems:

$$\begin{array}{|c|c|} \hline 2 & 4 \\ \hline 3 & 5 \\ \hline \end{array} = \begin{array}{|c|} \hline 8 \\ \hline \\ \hline \end{array}$$

Mr. Wilson told the class, "Alec's systems help explain what we mean by

Multiplication of Fractions.

"We can better understand the meaning of:

A. $\dfrac{2}{3} \times \dfrac{3}{4} = \text{---}$

if we can think of a situation in which you could find $\frac{2}{3}$ of something and then find $\frac{3}{4}$ of the result.

"Suppose we decide to think about a day (or 24 hours): $\frac{2}{3}$ of a day is _____ hours; and $\frac{3}{4}$ of this is _____ hours.

"Then we ask what fraction of a day gives us that final result.

"Alec would do the same thing:

a. $24 \boxed{\dfrac{2}{3}} \quad \boxed{\dfrac{3}{4}}$, $24 \boxed{\quad} 12$

$\boxed{\dfrac{2}{3} \ \dfrac{3}{4}} = \boxed{\quad}$

"How can we think about this?

B. $\dfrac{2}{5} \times \dfrac{1}{3} =$

"We can think of June (a 30-day month) and say:

"In the last month of school, only $\frac{2}{5}$ of the days are school days, and $\frac{1}{3}$ of the school days are incomplete school days.

"We multiply 30 by $\frac{2}{5}$ to find the number of school days, which is _____ days. Then we find $\frac{1}{3}$ of that which is _____ incomplete school days. What fraction of 30 gives us the same number as the number of incomplete school days?

"Alec would write:

b. $30 \boxed{2} \quad \boxed{1}$, $30 \boxed{\quad}$

"How can we think about this?

C. $\dfrac{5}{2} \times \dfrac{2}{3} =$

"At a department store, Mrs. Brown saw a piece of cloth she wanted. It was $2\frac{1}{2}$ yards long. She needed a piece that was 5 feet long. 'I wonder,' she said, 'if $\frac{2}{3}$ of that piece would be long enough.'

"First she thought of the fact that a half-yard is _____ inches. There are _____ half-yards in $2\frac{1}{2}$ yards. So, in the whole piece there are _____ inches of cloth. $\frac{2}{3}$ of that would be _____ inches, or _____ yards.

"Alec would write:

c. $36 \boxed{\quad} \quad \boxed{\quad}$, $36 \boxed{3}$

↑
Number of inches in the whole piece of cloth.

"How can we think about this?

D. $\dfrac{3}{4} \times 1\dfrac{1}{2} =$

"Mary had 40 books on her shelves. She said, 'I've read $\frac{3}{4}$ of them since school started.' Her sister replied, 'I've read $1\frac{1}{2}$ times as many as you have.'

"We might say that of Mary's 40 books, she had read $\frac{3}{4}$ of them, or _____ books. And $1\frac{1}{2}$ times 30 is _____. Then we want to know what fraction that is of 40 books.

"Alec might write:

d. $40 \boxed{\quad} \quad \boxed{\quad}$, $40 \boxed{\quad}$

"How can we think about these?

E. $\dfrac{1}{2} \times \dfrac{4}{5} =$ F. $\dfrac{3}{2} \times \dfrac{4}{9} =$
                             and

"Make up your own problems. Show how Alec might have solved them."

e. $\boxed{\dfrac{1}{2} \quad \dfrac{4}{5}}$, $\boxed{\quad}$

f. $\boxed{\dfrac{3}{2} \quad \dfrac{4}{9}}$, $\boxed{\quad}$

In the left column there are 11 questions selected from an achievement test.

Imagine you are Alec and wish to test each of the 4 possible answers given. You would select an appropriate input and try the 4-operation system and each 2-operation system. If inputs and outputs are the same, Alec would say that the systems are equal.

1.

$$\frac{2}{3} \times \frac{4}{3} = \Box$$

	A	B	C	D
	$\frac{5}{6}$	⟨$\frac{8}{9}$⟩	$\frac{6}{3}$	$\frac{8}{3}$

Draw a loop around the correct answer

	System	A	B	C	D
	$18\begin{array}{|c|c|}\hline 2 & 4 \\\hline 3 & 3 \\\hline\end{array}16$	$18\begin{array}{|c|}\hline 5 \\\hline 6 \\\hline\end{array}$	$18\begin{array}{|c|}\hline 8 \\\hline 9 \\\hline\end{array}$	$18\begin{array}{|c|}\hline 6 \\\hline 3 \\\hline\end{array}$	$18\begin{array}{|c|}\hline 8 \\\hline 3 \\\hline\end{array}$

So, Alec would select B as the correct answer.

2.

$$\frac{3}{4} \times \frac{2}{5} = \Box$$

	A	B	C	D
	$\frac{5}{10}$	$\frac{9}{5}$	$\frac{3}{10}$	$\frac{5}{20}$

	System	A	B	C	D
	$40\begin{array}{|c|c|}\hline 3 & 2 \\\hline 4 & 5 \\\hline\end{array}$	$40\begin{array}{|c|}\hline 5 \\\hline 10 \\\hline\end{array}$	$40\begin{array}{|c|}\hline 9 \\\hline 5 \\\hline\end{array}$	$40\begin{array}{|c|}\hline 3 \\\hline 10 \\\hline\end{array}$	$\begin{array}{|c|}\hline 5 \\\hline \\\hline\end{array}$

3.

$$\frac{9}{2} \times \frac{1}{3} = \Box$$

	A	B	C	D
	$\frac{10}{6}$	$\frac{3}{2}$	$\frac{11}{4}$	$\frac{7}{6}$

	System	A	B	C	D
	12				

4.

$$2\frac{1}{2} \times \frac{6}{10} = \Box$$

	A	B	C	D
	$\frac{10}{4}$	$\frac{12}{10}$	$\frac{3}{2}$	$\frac{8}{20}$

	System	A	B	C	D
	$20\begin{array}{|c|}\hline 5 \\\hline 2 \\\hline\end{array}$				

5.

$$\frac{1}{2} \times \frac{1}{2} = \Box$$

	A	B	C	D
	$\frac{1}{1}$	$\frac{2}{2}$	$\frac{2}{4}$	$\frac{1}{4}$

System	A	B	C	D

6.

$$\frac{3}{4} \times \frac{2}{3} = \Box$$

	A	B	C	D
	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{5}{6}$	$\frac{8}{3}$

System	A	B	C	D

7.

$$\frac{1}{4} \times \frac{7}{2} = \Box$$

	A	B	C	D
	$\frac{7}{6}$	$\frac{9}{4}$	$\frac{7}{8}$	$\frac{5}{3}$

System	A	B	C	D

8.

$$1\frac{1}{2} \times 1\frac{1}{2} = \Box$$

	A	B	C	D
	$\frac{5}{2}$	$\frac{9}{4}$	$\frac{3}{3}$	$\frac{3}{2}$

System	A	B	C	D

9.

$$2 \times \frac{5}{6} = \Box$$

	A	B	C	D
	$\frac{5}{3}$	$\frac{7}{6}$	$\frac{10}{12}$	$\frac{4}{3}$

System	A	B	C	D

10.

$$\frac{5}{6} \times 9 = \Box$$

	A	B	C	D
	$\frac{4}{1}$	$\frac{7}{3}$	$\frac{15}{2}$	$\frac{14}{6}$

System	A	B	C	D

11.

$$1\frac{2}{3} \times 4 = \Box$$

	A	B	C	D
	$\frac{20}{3}$	$\frac{8}{3}$	$\frac{3}{1}$	$\frac{1}{2}$

System	A	B	C	D

85

"What about

Division of Fractions?"

was Mabel's question to Mr. Wilson.

A.

"Let's remember," Mr. Wilson began, "that we can think of a division as undoing a multiplication.

"If Ben says that the model he wants costs $1.80, which is exactly 3 times as much as he has, what can you say about that situation?"

"Of course, Ben has _____ ¢."

"What actually happened was that Ben knew he had 60¢. He knew that 60¢ × 3 = $1.80. He talked about the results of a multiplication. To find out the amount he has, we need to undo the multiplication. We might say:

$$\$1.80 \ (\text{undo} \times 3) = \underline{\hspace{1cm}} ¢.$$

"Instead, we have a name for the procedure of undoing a multiplication. We call it 'division,' and indicate it with a sign: ÷ . So, we actually write:

$$\$1.80 \div 3 = \underline{\hspace{1cm}} ¢.$$

$$\div 3 \quad \text{means} \quad \text{undo} \times 3$$

B.

"Doris said she had read 40 pages of her book, which was $\frac{2}{3}$ of all the pages.

"She may have looked at the total number of pages and found that $\frac{2}{3}$ of that was 40 pages.

$$\underline{\hspace{1cm}} \text{pages} \times \frac{2}{3} = 40 \text{ pages.}$$

"We know only the results of the multiplication. To find the total number of pages, we simply undo the multiplication:

$$40 \text{ pages } (\text{undo} \times \tfrac{2}{3}) = \underline{\hspace{1cm}}$$

or, using a familiar shorthand:

$$40 \text{ pages} \div \tfrac{2}{3} = \underline{\hspace{1cm}} \text{ pages.}$$

$$\div \tfrac{2}{3} \quad \text{means} \quad \text{undo} \times \tfrac{2}{3}$$

"If 40 pages is $\frac{2}{3}$ of the total then 20 pages, $40 \div 2$, is one-third of the total; and 20 pages $\times\ 3$ is 60 pages — the total number of pages in the book."

$$\div \tfrac{2}{3} \quad \text{means} \quad \div 2, \times 3 \quad \text{or} \quad \times \tfrac{3}{2}$$

C.

"Baxter had 27 correct answers. He had $\frac{9}{10}$ or 9 out of every 10 questions correct.

27 answers $\left(\text{undo} \times \tfrac{9}{10}\right) = \underline{\hspace{1cm}}$ answers.

27 answers $\div \tfrac{9}{10} = \underline{\hspace{1cm}}$ answers.

"If $\frac{9}{10}$ of the total is 27 then $\frac{1}{10}$ of the total is $27 \div 9$ — or _____ answers. If that is $\frac{1}{10}$ then the total number of questions is 10 times as much: 3 pages $\times\ 10 = \underline{\hspace{1cm}}$ pages

$$\div \tfrac{9}{10} \quad \text{means} \quad \text{undo} \times \tfrac{9}{10}$$

$$\div \tfrac{9}{10} \quad \text{means} \quad \div 9, \times 10 \quad \text{or} \quad \times \tfrac{10}{9}$$

D.

"Helen paid $3.00 for a blouse. She said that that was half as much as she spent for her skirt.

"To find the cost of the skirt, we must undo $\times \tfrac{1}{2}$. We do that by $\div 1, \times 2$ or $\times —$.

"Her skirt cost her $_____."

E.

At lunch, Mr. Wilson told his family that they had already traveled $\frac{3}{4}$ of the way to the lake. They had gone 150 miles.

To find the distance from home to the lake, you must undo $\times —$.

You do that by $\div\ , \times$ or $\times —$.

So, the total distance must be _____ miles.

$\div \frac{2}{3}$ means to undo $\times \frac{2}{3}$

$\times \frac{3}{2}$ will do the job.

"So, we can write: $\div \frac{2}{3} = \times \frac{3}{2}$

"From our work with other examples on the last page, we can write:

$\div \frac{9}{10} = \times \underline{\quad}$ $\div \frac{1}{2} = \times \underline{\quad}$ $\div \frac{3}{4} = \times \underline{\quad}$

"Here are some of the experiments from Alec's notebook."

U means
undo the operations below.

1.

¢ | $\times 2$ / $\div 3$ | 10¢   or   10¢ | U $\times 2$ / $\div 3$ | ¢

"An amount of money multiplied by $\frac{2}{3}$ is 10¢. What is the amount of money — or, undo a multiplication by $\frac{2}{3}$."

2.
in. | $\frac{4}{9}$ | 16 in.   16 in. | U $\frac{4}{9}$ | in.

3.
$ | $\frac{5}{3}$ | $70 $70 | U $\frac{5}{3}$ | $

"All of the following examples are marked with Alec's 'U.' To save space he did not indicate the unit of measurements in the blue blocks.

4. 20 | U $\frac{1}{2}$ 5. 30 | U $\frac{2}{5}$ 6. 18 | U $\frac{3}{2}$

7. 36 | U $\frac{9}{1}$ 8. 45 | U $\frac{5}{6}$ 9. 28 | U $\frac{4}{7}$

"Alec added a step to help him remember:

10. 26 | U $\frac{\cancel{2}}{\cancel{3}}$ | $\div 2$ / $\times 3$

11. 5 | U $\frac{1}{\cancel{4}}$ | $\div$ / $\times$

"Alec added a third step to help him remember. Then he dropped the second step.

12. 20 | U $\frac{\cancel{5}}{\cancel{8}}$ | $\div 5$ / $\times 8$ | $\frac{8}{5}$

13. 35 | U $\frac{\cancel{7}}{\cancel{10}}$ | $\div$ / $\times 10$

14. 5 | U $\frac{\cancel{1}}{\cancel{8}}$ | $\frac{8}{1}$

15. 30 | U $\frac{\cancel{3}}{\cancel{5}}$ |

"Alec included his **U** notation in his study of systems.

A. 40 | $\frac{3}{4}$ | U $\frac{\cancel{5}}{\cancel{2}}$ | 2 , 40 | 12

B. 54 | $\frac{5}{6}$ | U $\frac{3}{2}$, 54

C. 20 | $\frac{7}{2}$ | U $\frac{5}{3}$, 20

D. 48 | $\frac{7}{8}$ | U $\frac{2}{3}$, 48

"Alec would then write:

a. $\frac{3}{4}$ | U $\frac{\cancel{5}}{\cancel{2}}$ = $\frac{3}{4}$ | $\frac{2}{5}$ = $\frac{3}{10}$

b. $\frac{5}{6}$ | U $\frac{\cancel{3}}{\cancel{2}}$ = $\frac{5}{6}$ | $\frac{2}{\quad}$ =

c. $\frac{7}{2}$ | U $\frac{5}{3}$ = $\frac{7}{\quad}$ | =

d. $\frac{7}{8}$ | U $\frac{2}{3}$ = | =

"And we would write:

a. $\frac{3}{4} \div \frac{5}{2} = \frac{3}{4} \times \underline{\quad} = \underline{\quad}$

b. $\frac{5}{6} \div \frac{3}{2} = \underline{\quad} \times \underline{\quad} = \underline{\quad}$

c. $\frac{7}{2} \div \frac{5}{3} = \underline{\quad} \times \underline{\quad} = \underline{\quad}$

d. $\frac{7}{8} \div \frac{2}{3} = \underline{\quad} \times \underline{\quad} = \underline{\quad}$

"Some people call this procedure for dividing by a fraction the *Invert and Multiply Rule*."

I.
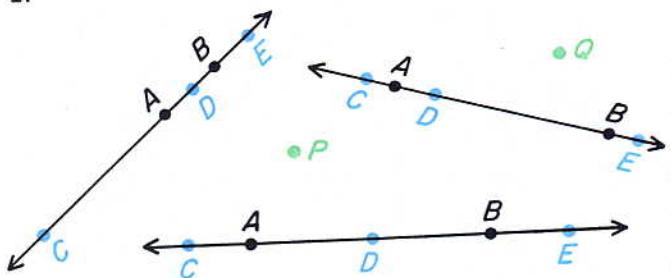
Lines are drawn through each of three pairs of points labeled A and B. The arrows indicate that we are thinking of straight lines (extending indefinitely).

On each line, three additional points have been labeled C, D, and E. Two other points not on the lines are labeled P and Q.

WHAT CAN YOU SAY that is true about all three of these lines?

If a statement below is *true about all three lines,* put a loop around T; if it is not true about all three, put a loop around F.

(1) The distance from C to A is less than the distance from A to D. T (F)

(2) The distance from A to C is less than the distance from C to D. T F

(3) D is half-way between A and B. T F

(4) A is between C and D. T F

(5) Both D and B are between C and E. T F

(6) The distance from E to B is less than the distance from B to D. T F

(7) A, B, and D are all located between C and E. T F

(8) D is not between A and C. T F

(9) If all three lines drawn were extended, each would cross the other two. T F

(10) If all three lines drawn were extended, a triangle would be formed. T F

(11) Both P and Q would be inside that triangle. T F

(12) A line drawn through P and Q (extended) would intersect (or cross) all three lines drawn through A and B (extended). T F

II.
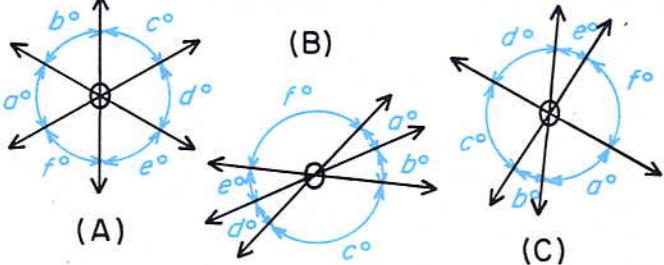
In each sketch, three straight lines intersect at a point labeled O.

WHAT CAN YOU SAY that is true about all three sketches?

(1) $a = c$ T F

(2) $b + c + d = 180$ T F

(3) $c > a$ (c is greater than a) T F

(4) $c < a$ (c is less than a) T F

(5) $a = d$ T F

(6) $b + c = e + f$ T F

(7) $b - e = a - d$ T F

(8) $f = c$ and $e = b$ T F

(9) $a + b = d + e$ T F

(10) $a + b + c + d + e + f = 360$ T F

(11) $a + d = f + c$ T F

III.
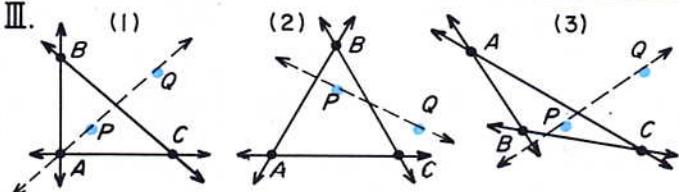
In each sketch, the three lines intersect at points labeled A, B, and C; also P and Q are on a fourth line.

WHAT CAN YOU SAY that is true about all three sketches?

(1) The line PQ intersects the line BC. T F

(2) The distance between B and C is greater than the distance between A and C. T F

(3) The distance from A to B plus the distance from B to C is greater than the distance from A to C. T F
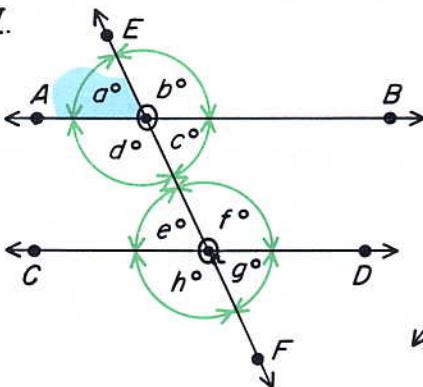
Line AB is parallel to line CD. Line EF intersects line AB at O and CD at Q.

WHAT CAN YOU SAY that is true about all three sketches?

(1)	O is between A and B.	(T)	F
(2)	$d = 90$	T	F
(3)	$a = c$	T	F
(4)	$d = h$	T	F
(5)	$a + b = a + d$	T	F
(6)	$b > a$ (b is greater than a)	T	F

List some of your own statements that are true about all 3 sketches above.

— — — — — — — — — — — — — —

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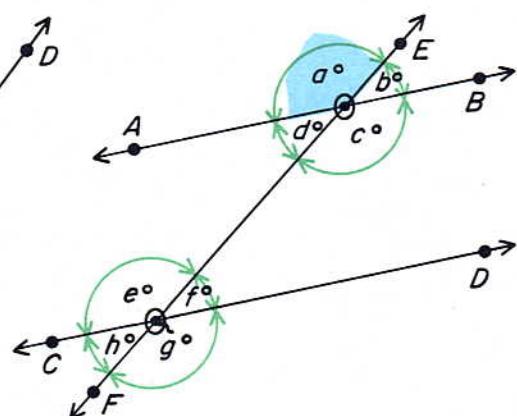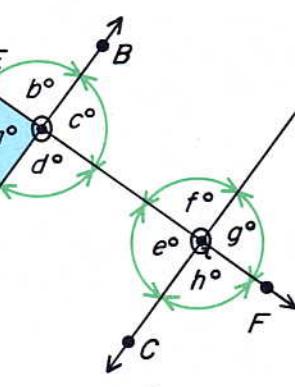
The angle of $a°$ formed by lines AO and EO can be referred to as $\angle AOE$ (read: angle AOE) or as $\angle EOA$.

(7)	$\angle CQF \cong \angle OQD$	T	F
(8)	$b = h$	T	F
(9)	$a + b + h + g = 180$	T	F
(10)	$c + h = 180$	T	F
(11)	$\angle AOQ \cong \angle OQD$	T	F
(12)	$\angle FQD \ncong$ (is not congruent to) $\angle AOE$		
		T	F

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(1)	$r = 90$	(T)	F
(2)	$\angle ABC$ is a right angle.	T	F
(3)	The length of AC is greater than the length of BC.	T	F
(4)	$r + s + t = 180$	T	F
(5)	$r + s > 130$	T	F
(6)	$s + t = 90$	T	F
(7)	$r - s = t$	T	F
(8)	$r > t$	T	F
(9)	$s > t$	T	F
(10)	$r > s$	T	F
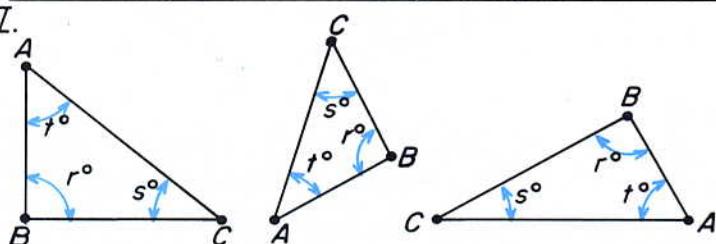
The line segments AB and BC are perpendicular — they form a right angle.

WHAT CAN YOU SAY about one of the triangles that is true for all of them?

(11)	$s = t$	T	F

List some of your own statements that are true for all 3 sketches.

— — — — — — — — — — — — — — —

— — — — — — — — — — — — — — —

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89
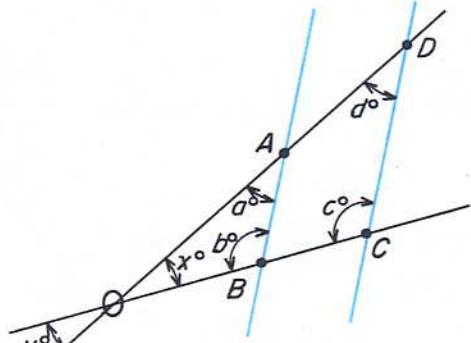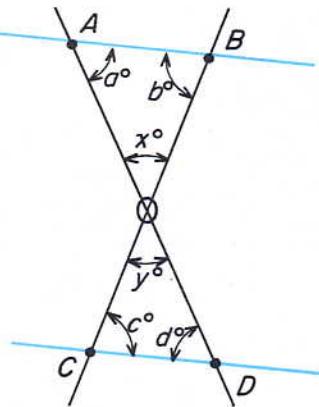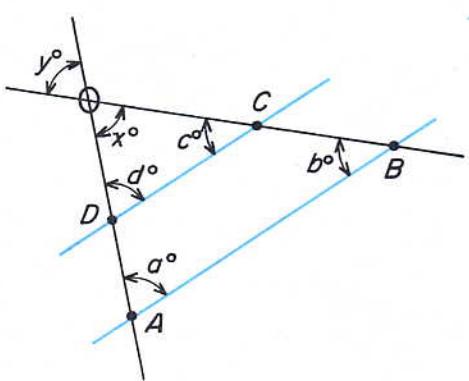
I. In each sketch we have drawn a pair of parallel lines. We have drawn two more lines that intersect each other at O. One of these two lines intersects the parallel lines at A and D; the other intersects them at B and C.

> WHAT CAN WE SAY that applies to all of the sketches?

> If the angles of one triangle are the same size as the angles of another triangle, we say the triangles are SIMILAR — they have the same shape. A backward "S" on its side is shorthand for the idea.
>
> "$\sim$" means *is similar to*

(1) $a = d$ T F

(2) $d = b$ T F

(3) $x = c$ T F

(4) $b = c$ T F

(5) $x = y$ T F

(6) $a + b = c + d$ T F

(7) $\triangle ODC \sim \triangle OAB$ T F

(8) $x + d + c = x + a + b$ T F

(9) $\triangle BOA$ is the same size and shape as $\triangle COD$. T F

II.   SAS
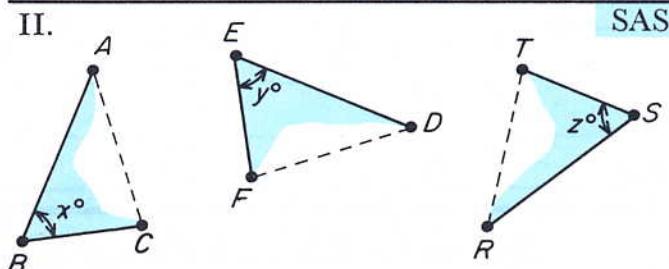
> IF sides AB, DE, and RS are the same length; and if sides BC, EF, and ST are the same length; and if $x = y = z$,
>
> THEN What Can You Say?

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- - - - - - - - - - - - - - - - - - - -

ASA

III.
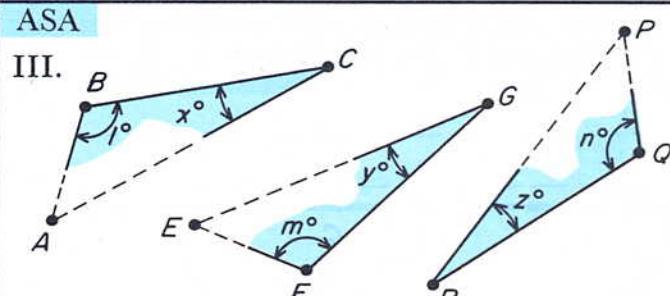
> IF sides BC, FG, and RQ are the same length, and if $l = m = n$ and $x = y = z$,
>
> THEN What Can You Say?

- - - - - - - - - - - - - - - - - - - -

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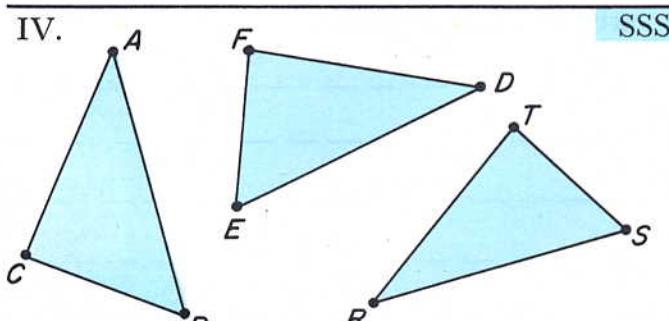
IV.   SSS

> IF sides AB, DE, and RS are the same length; and if sides BC, EF, and ST are the same length; and if sides AC, DF, and RT are the same length,
>
> THEN What Can You Say?

- - - - - - - - - - - - - - - - - - - -

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"$\cong$" means *is congruent to* or *is the same size and shape as*

90
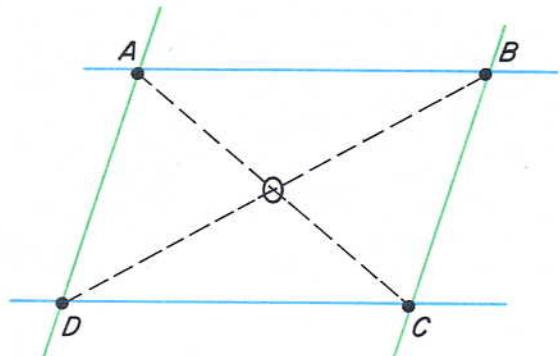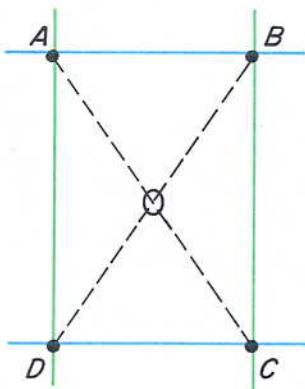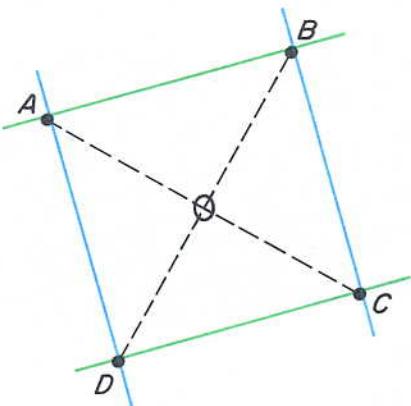
A pair of parallel lines (blue) intersects another pair of parallel lines (green). The points of intersection are labeled A, B, C, and D. Diagonals have been drawn connecting A and C, and connecting B and D. The intersection of the diagonals is labeled O. - - - - - - - ->

Note: The 4-sided figure created when two pairs of parallel lines intersect — such as the figures we have labeled $ABCD$ — are called PARALLELOGRAMS.

All squares and rectangles are usually considered as special cases of parallelograms.

What Can You Say?

(1) $ABCD$ is a parallelogram.   T   F

(2) $ABCD$ is a square.   T   F

(3) $ABCD$ is a rectangle.   T   F

(4) The distance between A and B is the same as the distance between B and C.   T   F

(5) The distance between A and D is the same as the distance between B and C.   T   F

(6) The diagonal AC has the same length as the diagonal BD.   T   F

(7) The distance between A and O is the same as the distance between O and C.   T   F

(8) $\angle ABC$ is a right angle.   T   F

(9) $\angle DOC$ is the same size as $\angle DOA$.   T   F

(10) $\triangle ABC$ is a right triangle.   T   F

(11) $\triangle ABC$ is the same size and shape as $\triangle CDA$.   T   F
(If 2 figures have the same size and shape, one could be cut out and put down on the other with all the points of one fitting the points of the other exactly.)

(12) $\triangle ABD$ is the same size and shape as $\triangle BCA$.   T   F

(13) $\angle BAO$ is the same size as $\angle DCO$.   T   F

(14) $\angle ADC$ is the same size as $\angle ABC$.   T   F

(15) The distance between D and O is the same as the distance between O and B.   T   F

(16) $\triangle ABO$ is the same size and shape as $\triangle CDO$.   T   F

(17) $\angle BOC$ is the same size as $\angle AOB$.   T   F

(18) $\triangle AOD$ is the same size and shape as $\triangle COB$.   T   F

(19) $\triangle AOB$ is not the same shape as $\triangle AOD$, but it is the same size.   T   F

Consider statement (19) further. Let's redraw those triangles from the sketch on the right:
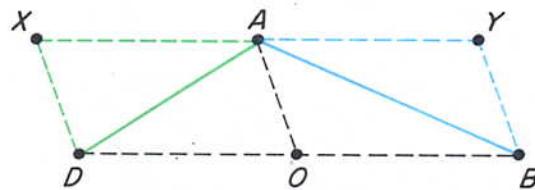
We draw DX and YB parallel to AO and XA and AY parallel to DB. Remember that the sides DO and OB have the same length (15).

(20) $AODX$ and $YBOA$ are parallelograms of the same size and shape.   T   F

(21) $\triangle ADO$ and $\triangle DXA$ have the same size and shape.   T   F

Fig. I   1 unit of area

Fig. II   1 unit of area

Fig. III   1 unit of area

Fig. IV   1 unit of area

Fig. V   1 unit of area

Fig. VI   1 unit of area

Fig. VII   1 unit of area

Fig. VIII   1 unit of area
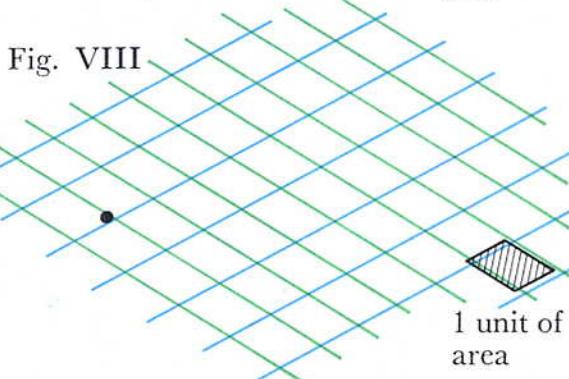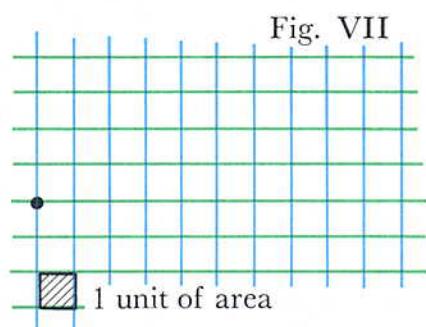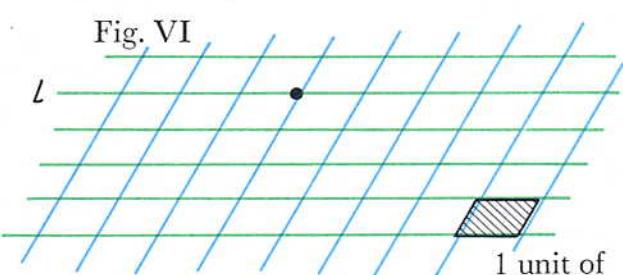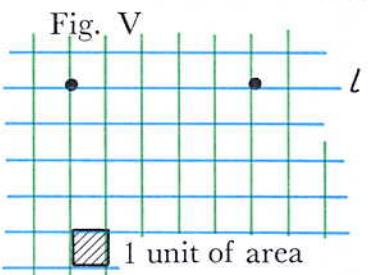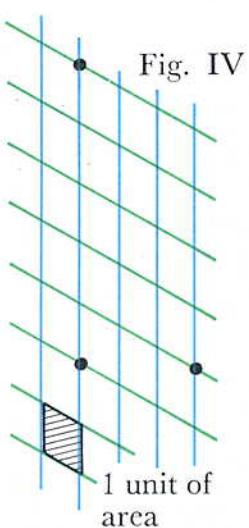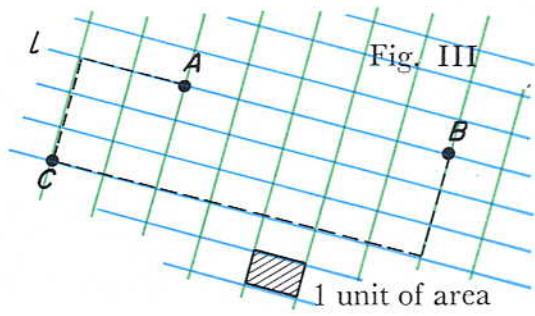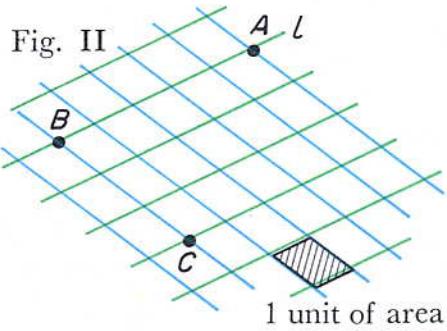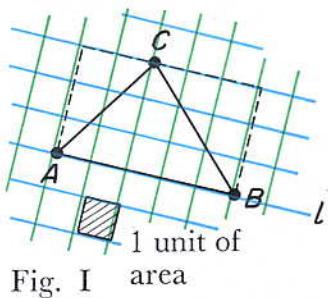
A set of equally spaced parallel lines is drawn to intersect another set of equally spaced parallel lines.

A line is selected and labeled l.

On l, a point A is located where a line intersects l. Also on l, a point B is located 5 spaces from point A.

A point C is located at any intersection on a line parallel to l and 3 spaces from l.

Lines are drawn connecting points A and B, points B and C, and points A and C.

(Only Figure I is completed according to the description above. Please complete the other figures by following the description.)

(Additional broken lines are drawn in Figures I and III that may help you with the next question.)

What can you say about $\triangle ABC$ that is true with regard to all the figures?

— — — — — — — — — — — —

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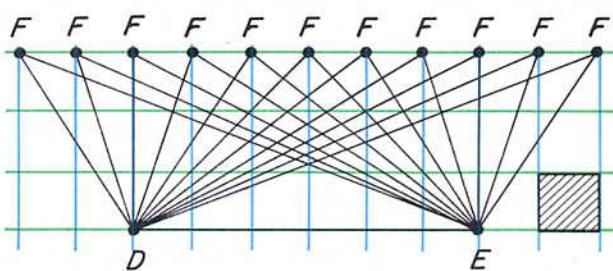
What can you say about the area of triangles labeled DEF?

— — — — — — — — — —

— — — — — — — — — —

I.
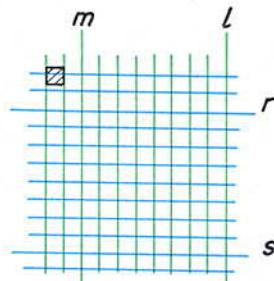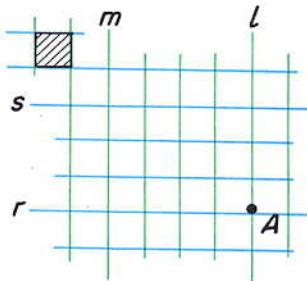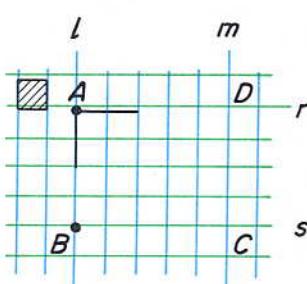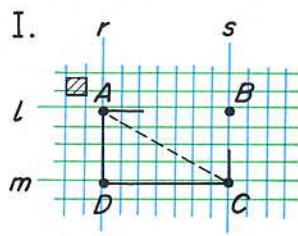
Follow the directions and complete the sketches.

1. One set of equally spaced parallel lines intersects another set with the same equal spacing at right angles.

2. Two lines in one set are selected and labeled r and s. Two lines in the other set are selected and labeled l and m.

3. The intersection of l and r is labeled A, of l and s is labeled B, of s and m is labeled C, of r and m is labeled D. A unit of area is indicated.

4. Draw sides AB, BC, CD, and DA and diagonal AC.

(1) $ABCD$ is a rectangle.   T   F

(2) $ABCD$ is a parallelogram.   T   F

(3) The side AB is longer than the side BC.   T   F

(4) There are as many equal spaces between A and B as between C and D.   T   F

(5) The number of units of area of rectangle $ABCD$ is the same as the number of spaces between A and B multiplied by the number of spaces between A and D.   T   F

(6) The number of units of area in rectangle $ABCD$ is twice the number of units of area in triangle ABC and twice as many as in triangle ADC.   T   F

(7) $\triangle ABC \cong \triangle CDA$ T   F

II.
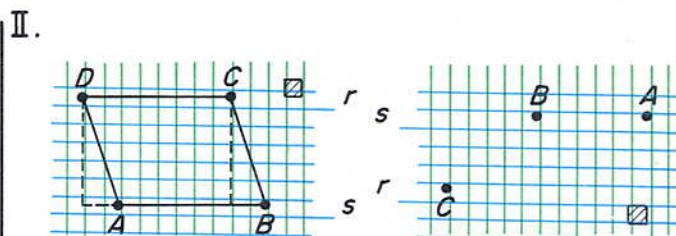
Follow the directions and complete the sketches.

1. Two sets of parallel lines with the same equal spacing intersect.

2. Points A and B are located on line s. On a line parallel to s and labeled r, two points, D and C, are located the same number of equal spacings apart as A and B — labeled so that lines AD, DC, CB, and BA form a parallelogram.

3. Draw line AC. Draw broken lines from D and C perpendicular to line s. Label these lines DM and CN. Connect M and A and connect N and B (unless they are already connected).

(1) $\triangle ABC \cong \triangle CDA$ T   F

(2) $\triangle DMA \cong \triangle CNB$ T   F

(3) The distance between B and M is greater than the distance between A and N.   T   F

(4) $ABCD$ is a rectangle.   T   F

(5) $DCNM$ is both a rectangle and a parallelogram.   T   F

(6) The number of area units in rectangle $MNCD$ is the number of spaces between points A and B multiplied by the number of spaces between lines r and s.   T   F

(7) The number of area units in $\triangle DMA$ is the same as the number of area units in $\triangle CNB$.   T   F

(8) The area of $\triangle ADC$ equals the area of $\triangle ACB$.   T   F

(9) The area of $\triangle ACB$ is the sum of the areas of $\triangle ACN$ and $\triangle CNB$.   T   F

(10) The area of $\triangle ADC$ is the sum of the areas of $\triangle ACN$ and $\triangle DMA$.   T   F

(11) The area of $\triangle ADC$ is half the area of parallelogram $ABCD$ and is also half the area of rectangle $CDMN$.   T   F

(12) $DCNA$ is a parallelogram.   T   F

I.

II.

On a line locate two points, A and B.

On a line parallel to line AB locate four points and label them D, E, F, and G.

Draw triangles ABD, ABE, ABF, and ABG.

What can you say about the areas of the four triangles in any situation like that?

_____

_____

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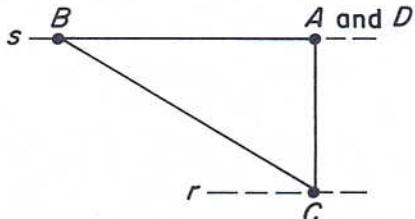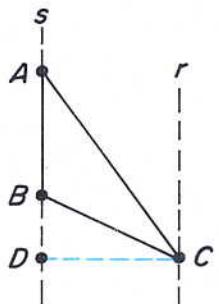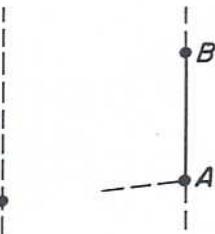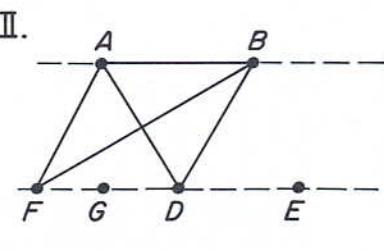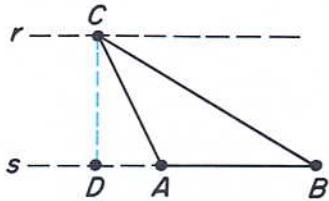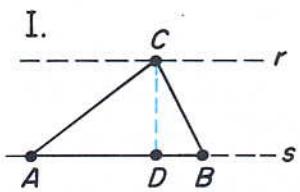
Two sets of equally spaced parallel lines intersect at right angles. From one set, lines r and s are selected. Points A and B are located at intersections on s. Point C is located at an intersection on line r. A line from C is drawn perpendicular to s. It meets s at a point labeled D. (In special cases, points A and D may be the same.)

Draw triangle ABC.

What can you say about the area of $\triangle ABC$ in all of the sketches?

The number of spaces between r and s (or C and D) multiplied by the number of spaces between A and B is twice the number of area units in $\triangle ABC$.   T   F

The number of spaces between r and s (or between C and D) is called the *Altitude* of the triangle, and the number of spaces between A and B is called the *Base* of the triangle.

Let a be the altitude of a triangle and b be the base. Then:

> Area of a triangle $= \frac{1}{2} a \times b$ or $\frac{1}{2} ab$.

III.

Locate points A and B on a line, and C and D on a line parallel to line AB, so that connecting A to D and B to C will create a quadrilateral with diagonals AC and BD crossing at O.

What can you say about the areas of $\triangle ABD$ and $\triangle ABC$? . . . and about the areas of $\triangle AOD$ and $\triangle BOC$?

_____

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What can you say about the shapes of $\triangle AOB$ and $\triangle COD$?

_____

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_____

I.
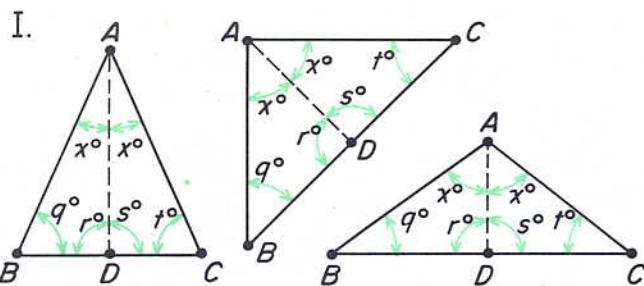
These triangles are drawn so that, in each, sides AB and AC have the same length.

A line is drawn from A to the point labeled D on the opposite side; and line AD is drawn so that it bisects $\angle BAC$ — so $\angle BAD$ is the same size as $\angle CAD$.

(1) Sides AB and AC have the same length.   T   F

(2) $\triangle ADB \cong \triangle ADC$ T   F

(3) $r = s = 90$ T   F

What else can you say that is true about all three figures?

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _

If a triangle has two sides of equal length, we call it an ISOSCELES triangle.

Because $q = t$ above, we say that in an isosceles triangle the angles opposite the sides of equal length are equal in size.

II.
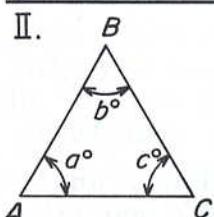
Sides AB, AC, and BC are all equal in length. If the three sides of a triangle are the same length, we call that an EQUILATERAL triangle.

What can you say about the angles of an equilateral triangle?

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _

III.
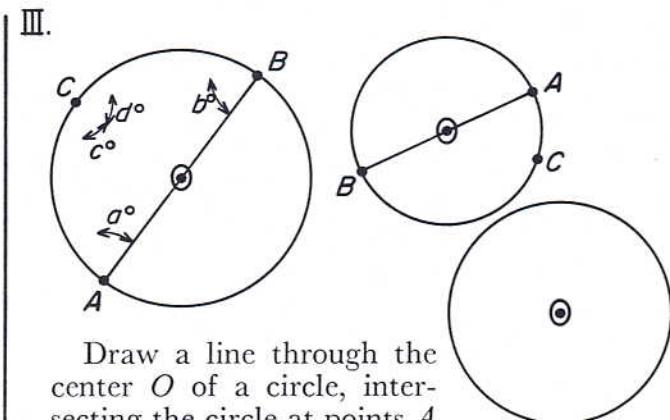
Draw a line through the center O of a circle, intersecting the circle at points A and B. Locate a third point on the circle and label it C. Draw triangle ABC. Draw line OC.

What can you say for all the sketches above?

(1) The distance between A and O is the same as the distance between O and B.   T   F

(2) Point O is just as far from A as it is from C.   T   F

(3) Point C is just as far from A as it is from B.   T   F

(4) $a = c$ T   F

(5) $a = b$ T   F

(6) $b = d$ T   F

(7) $a + b = c + d$ T   F

(8) $a + b + c + d = 180$ T   F

(9) $a + b = 100$ T   F

(10) $c + d = 90$ T   F

(11) $\triangle AOC \cong \triangle BOC$ T   F

(12) The area of $\triangle AOC$ is the same as the area of $\triangle BOC$.   T   F

(13) Angle ACB is a right angle.   T   F

(14) $a + d = c + b$ T   F

IV.
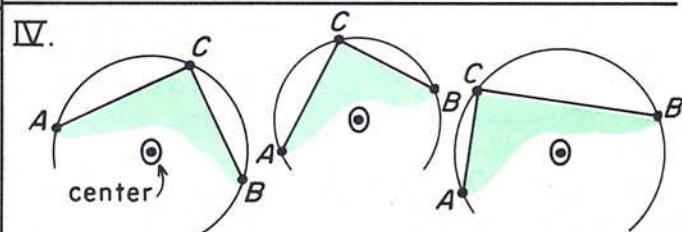
What can you say about line AB if $\angle ACB$ is a right angle?

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _
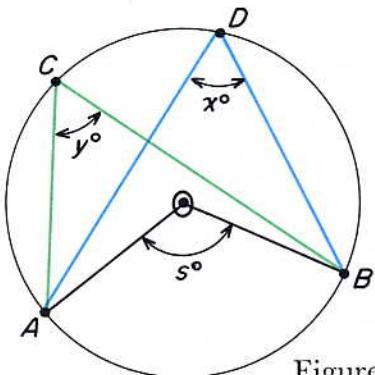
Figure 1
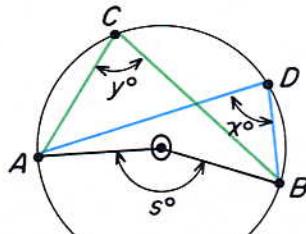
Figure 2
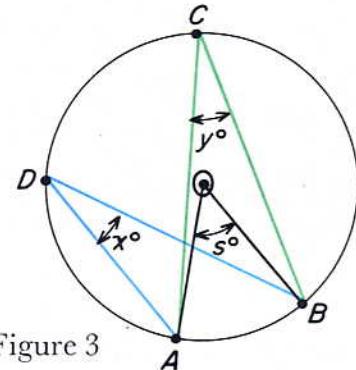
Figure 3

Points A and B are located on a circle, and a straight line is drawn from each to the circle's center, labeled O.

Two other points on the circle are labeled C and D. Lines from A and B are drawn to C and to D.

What can you say that's true about all the figures above?

Before making any statements, use a protractor to find y, x, and s in each figure and tabulate your results in the chart in the next column.

Here is another approach — a test of your conclusions based on experiments.

We shall consider $\angle AOB$ and $\angle ADB$ in Figure 1 above.
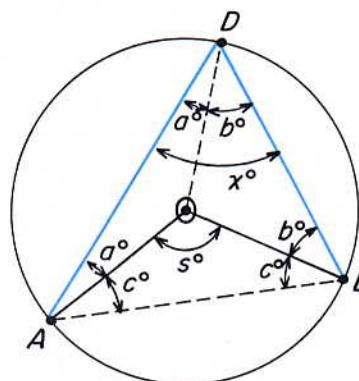
We draw lines connecting D and O, and connecting A and B.

Sides AO, BO, and DO are all the same length.

$\triangle DOA$, $\triangle AOB$, and $\triangle DOB$ are all isosceles triangles.

So, angles OAD and ODA are the same size; so are angles ODB and OBD and so are angles OBA and OAB.

$$(c + a) + (a + b) + (b + c) = 180$$

or .... $2a + 2b + 2c = 180$.

But .... $s + 2c = 180$.

So .... $s = 2a + 2b = 2x$.

Number of Degrees

	x	y	s
Fig. 1			
Fig. 2			
Fig. 3			

My conclusions are:

Does the argument on the left strengthen your belief in the conclusions you based on experiments?

However, do not jump to the conclusion that our argument is a complete proof. One of the examples above cannot be handled in the same way. Consider $\angle ABD$ and $\angle AOB$ in Figure 3.
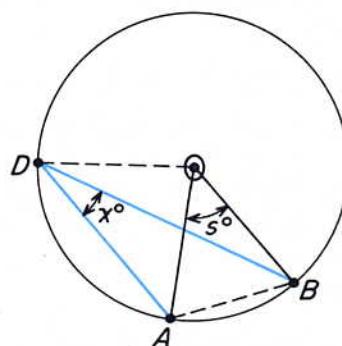
Again, draw sides OD and BA.

Notice that sides OD, OA, and OB are all the same length.

Thus, angles OAD and ODA are the same size; so are angles ODB and OBD and so are angles OBA and OAB.

Try to proceed with an argument that will support the conclusion that $2x = s$.

Measurement Reports in Decimal Fractions

Primitive man may have invented units of length before inventing units for other quantities, using the length of his foot, the width of his hand, and, for finer measurement, the width of his thumb.

Man's need for fine measurement has kept pushing his ability to devise better instruments.

When automobile racing began in 1897, the record speed was reported to be 37.5 miles per hour. In 1965, when Craig Breedlove broke the record at Bonneville Flats, his average speed over a 2-mile course was 600.601 m.p.h. — 600 and 601 thousandths of a mile per hour.

Experience in the annual coast-to-coast Mobil Gas Economy Run has required reporting results in hundredths of miles per gallon.

Here are part of the results of one of the annual runs:

Intermediate Size V-8's

	Miles per gallon
Dodge Coronet	20.76
Plymouth Belvedere	20.61
Buick Special	20.30
Oldsmobile F85	19.80
Comet 404	19.58
Ford Fairlane	19.28
Pontiac Tempest	19.11
Rambler Ambassador	18.44
Chevelle Malibu	18.17

What Can You Say? _____

_____

Here are the cars that placed first, second, and third in their class:

Fullsize Medium Priced V-8's

	miles per gallon
Pontiac Catalina	19.84
Pontiac Star Chief	19.64
Buick LeSabre	19.63

What Can You Say? _____

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In the 1965 professional baseball leagues, pitchers were ranked on the basis of the average number of earned runs they allowed in each 9 innings they pitched.

McDowell of Cleveland won top honors for the lowest earned run average (E.R.A.) of pitchers in the American League who pitched more than 200 innings — an E.R.A. of only 2.17 earned runs in each 9 innings.

Koufax of Los Angeles had the best record in the National League — an E.R.A. of only 2.04 for 9 innings.

What Can You Say? _____

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On February 26, 1966, the Weather Bureau in Monterey, California, reported the amount of rainfall for the last 24 hours, the amount since July 1, 1965, and the total amount between July 1, 1964 and February 26, 1965. Here is part of the report. (The measurements are in inches.)

	Last 24 hrs.	This year	Last year
Carmel	.20	16.17	12.10
Marina	.30	14.82	12.13
Monterey	.23	16.69	14.22
Pacific Grove	.19	16.49	12.58
Salinas	.20	11.81	9.32
St. Clemente	.12	14.77	13.56

What Can You Say? _____

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On June 3, 1960, Captain Yates flew a Boeing 707 from New York to San Francisco at the rate of 510.092 m.p.h. On December 8, 1961, Captain Miller flew a Boeing 707 from Los Angeles to New York at the rate of 636.8 m.p.h. Both were records.

What Can You Say? _____

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Without the shorthand we call "decimal fractions," we would have had to use expressions such as these:

$$600\frac{601}{1000} \qquad 20\frac{76}{100} \qquad 636\frac{8}{10} \text{ etc.}$$

instead of:

$$600.601 \qquad 20.76 \qquad 636.8$$

This shorthand uses the idea of place value to the right of units or ones place. Here's the plan:

tens ⟶
hundreds ⟶
units
⟶ tenths
⟶ hundredths
⟶ thousandths

$$5\ 1\ 0\ .\ 0\ 9\ 2$$
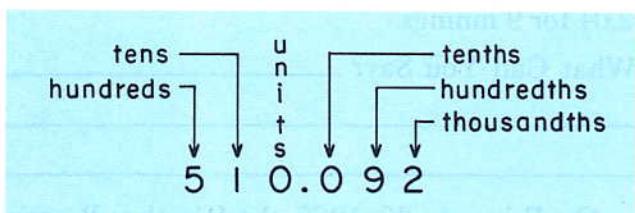
Read it as five hundred ten and ninety-two thousandths.

As we learn more about this shorthand, we shall discover important shortcuts.

Fractions already written as tenths, hundredths, thousandths, etc., can be written directly in decimal fraction shorthand. Here are some examples:

$$\frac{1}{10} = .1 \qquad \frac{8}{10} = \qquad \frac{6}{10} =$$

$$\frac{17}{100} = .17 \qquad \frac{10}{100} = \qquad \frac{57}{100} =$$

$$\frac{9}{100} = .09 \qquad \frac{1}{100} = \qquad \frac{70}{100} =$$

$$\frac{9}{1000} = .009 \qquad \frac{11}{1000} = \qquad \frac{125}{1000} =$$

$$\frac{5}{10} = \qquad \frac{5}{100} = \qquad \frac{5}{1000} =$$

$$\frac{9}{10} = \qquad \frac{75}{100} = \qquad \frac{36}{1000} =$$

$$\frac{8}{100} = \qquad \frac{83}{1000} = \qquad \frac{80}{1000} =$$

$$1\frac{6}{10} = 1.6 \qquad 7\frac{3}{10} = \qquad 2\frac{18}{100} = 2.18$$

$$3\frac{8}{10} = \qquad 3\frac{8}{100} = \qquad 6\frac{235}{1000} =$$

$$125\frac{9}{10} = \qquad 7\frac{8}{100} = 7.08 \qquad 7\frac{8}{1000} =$$

$$40\frac{7}{10} = \qquad 40\frac{7}{100} = \qquad 40\frac{70}{1000} =$$

Some fractions can easily be rewritten as tenths, hundredths, thousandths, etc., and then written in decimal fraction shorthand. Here are some examples:

$$\frac{1}{2} = \frac{5}{10} = .5 \qquad \frac{1}{4} = \frac{25}{100} = .25$$

$$\frac{1}{5} = \frac{}{10} = \qquad \frac{3}{4} = \frac{}{100} =$$

$$\frac{3}{5} = \frac{}{10} = \qquad \frac{2}{5} = \frac{}{10} =$$

$$\frac{1}{20} = \frac{}{100} = \qquad \frac{1}{25} = \frac{}{100} =$$

$$\frac{1}{50} = \frac{}{100} = \qquad \frac{4}{5} = \frac{}{} =$$

$$\frac{3}{20} = \frac{}{100} = \qquad \frac{3}{25} = \frac{}{} =$$

$$\frac{3}{50} = \frac{}{} = \qquad \frac{7}{20} = \frac{}{} =$$

$$\frac{1}{8} = \frac{125}{1000} = \qquad \frac{3}{8} = \frac{}{} =$$

$$\frac{7}{8} = \frac{}{} = \qquad \frac{5}{8} = \frac{}{} =$$

Some fractions can be reduced and then written as decimal fractions.

$$\frac{2}{4} = \frac{1}{2} = .5 \qquad \frac{3}{12} = \frac{}{4} = .25$$

$$\frac{12}{16} = \frac{}{4} = \qquad \frac{7}{14} = \frac{}{} =$$

$$\frac{3}{15} = \frac{}{} = \qquad \frac{9}{36} = \frac{}{} =$$

$$\frac{2}{40} = \frac{}{} = \qquad \frac{12}{30} = \frac{}{} =$$

$$\frac{3}{75} = \frac{}{} = \qquad \frac{18}{24} = \frac{}{} =$$

$$\frac{28}{35} = \frac{}{} = \qquad \frac{3}{24} = \frac{}{} =$$

$$\frac{18}{36} = \frac{}{} = \qquad \frac{48}{50} = \frac{}{} =$$

$$\frac{33}{44} = \frac{}{} = \qquad \frac{25}{40} = \frac{}{} =$$
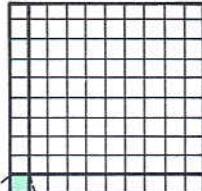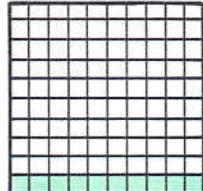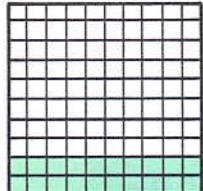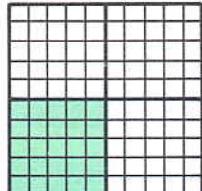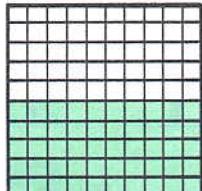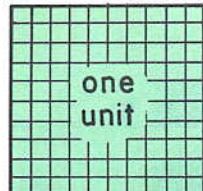
$1 = \frac{10}{10} = \frac{100}{100}$ $\frac{1}{2} = \frac{5}{10} = \frac{50}{100}$ $\frac{1}{4} = \frac{25}{100}$ $\frac{1}{5} = \frac{2}{10} = \frac{20}{100}$ $\frac{1}{10} = \frac{10}{100} = .1 = .10$ $\frac{1}{100}$

$1 = 1.0 = 1.00$ $.5 = .50$ $.2 = .20$
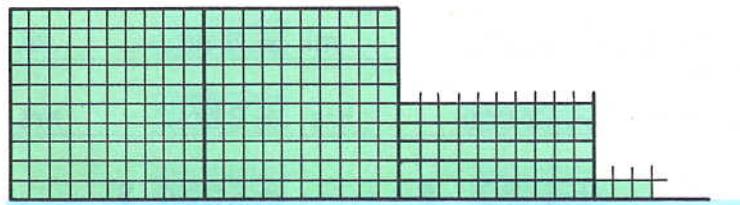
$2 + .5 + .03 = 2.53$

Two _and_ fifty-three hundredths

As we read decimal shorthand the word "and" signals the decimal point at the right of the units place.

What Are My Rules?

	3	15	1
x 100	300	1,500	
x 10	30		
x 1			
÷ 10	.3		
÷ 100			

	.4	8.7	.25
x 100	40		
x 10			
x 1			
÷ 10			
÷ 100			

1.7	x 3	5.1	÷ 100	.051
25	x 15		÷ 100	
.18	x 4		÷ 10	

one hundredth
(under a magni-
fying glass)

$\frac{1}{100} = \frac{10}{1000} = \frac{100}{10,000}$

$.01 = .010 = .0100$

one thousandth

$\frac{1}{1000} = \frac{10}{10,000}$

$.001 = .0010$

one ten thousandth

$\frac{1}{10,000}$ or .0001

Chain Reactions

I.

(a) $1.50 | +7 dimes | $2.20 | − 37¢ | $ | +half a dollar | $ | + $.80 | $

(b) 7.5 miles | + .6 mi. | mi. | + .9 mi. | mi. | − .7 mi. | mi. | + .25 mi. | mi.

(c) 5.25 in. | + .5 in. | in. | − 3 in. | in. | + .6 in. | in. | − 0.5 in. | in.

(d) $ .70 | + $.45 | $ | − 25¢ | $ | + $7 | $ | + 1 dime | $

(e) .75 miles | − .05 mi. | mi. | + .3 mi. | mi. | − .37 mi. | mi. | + .9 mi. | mi.

(f) 2 hours | + 1.5 hrs. | hrs. | − 1.7 hrs. | hrs. | − .25 hrs. | hrs. | + 8 hrs. | hrs.

(g) $ 3.00 | − $1.83 | $ | − 50¢ | $ | + $.90 | $ | + $1.57 | $

II. The new family car had gone only $17\frac{1}{2}$ miles before the trip began. Figure out the distance between each town and the next.

The shortest distance between two towns was _____ miles. The longest distance between neighboring towns was _____ miles. The total length of the trip was _____ miles.

Tens of thousands of miles / Thousands of miles / Hundreds of miles / Tens of miles / Unit miles / Tenths of a mile

						Town	Distance in miles between towns
0	0	0	1	7	5	Home	
							10.0
0	0	0	2	7	5	Malta	
0	0	0	4	7	9	Troy	
0	0	0	5	0	0	Dover	
0	0	0	8	5	2	Union	
0	0	0	9	9	9	Lee City	
0	0	1	0	8	0	Bureau	
0	0	1	7	5	3	Olan	
0	0	2	0	0	0	Benton	
0	0	2	6	7	8	Rapids	
0	0	2	7	7	7	Knox	
0	0	3	0	1	3	Hull	
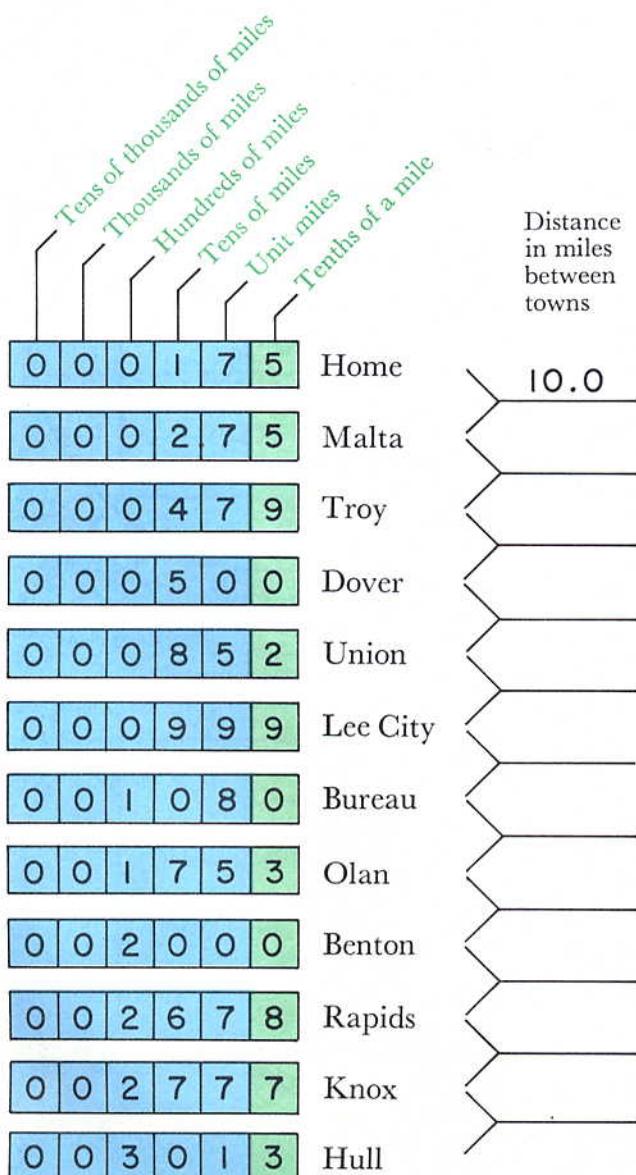
Distance between			(in miles)
Malta	and	Dover	22.5
Dover	and	Lee City	
Lee City	and	Benton	
	and		58.0
	and		101.3
	and		32.5
	and		125.3
	and		157.8
	and		57.7
	and		250.2
	and		192.5

A round trip between Olan and Benton is _____ miles. A round trip between _____ and _____ is 29.4 miles. Half of the distance between _____ and _____ is 33.9 miles.

In _____ more miles, the speedometer will read 1000.0.

100

I. A Different Kind of Chain Reaction.

(a) $ 2.50 ⟦x 3⟧ $ _____ ⟦÷10⟧ $ _____

(b) 8 in. ⟦x 9⟧ _____ in. ⟦÷100⟧ _____ in.

(c) 1.5 hrs. ⟦x 12⟧ _____ hrs. ÷10 _____ hrs.

(d) 1.2 miles ⟦x 12⟧ _____ mi. ÷100 _____ mi.

(e) $ 415 ⟦x 5⟧ $ _____ ⟦÷100⟧ $ _____

(f) .3 in. ⟦x 3⟧ _____ in. ÷10 _____ in.

(g) .7 min. ⟦x 13⟧ _____ min. ÷100 _____ min.

(h) $ 24.25 ⟦x 6⟧ $ _____ ⟦÷100⟧ $ _____

In Alec's style:

(i) 8 lbs. ⟦x 7 / ÷10⟧ _____ lbs.

(j) 12 ft. ⟦x 5 / ÷10⟧ _____ ft.

(k) 1.8 mi. ⟦x 8 / ÷10⟧ _____ mi.

(l) $ .05 ⟦x 9 / ÷10⟧ $ _____

(m) 16 hrs. ⟦x 12 / ÷10⟧ _____ hrs.

(n) 16 hrs. ⟦x 12 / ÷100⟧ _____ hrs.

(o) 13.5 in. ⟦x 15 / ÷100⟧ _____ in.

(p) $ 70.28 ⟦x 3 / ÷100⟧ $ _____

(q) 1.6 min. ⟦x 16 / ÷10⟧ _____ min.

(r) $ 125 ⟦x 9 / ÷100⟧ $ _____

(s) .3 mi. ⟦x 3 / ÷100⟧ _____ mi.

(t) $ 765 ⟦x 1 / ÷1000⟧ $ _____

II. Multiplying fractions:

(a) $7 \times \frac{1}{10} = \frac{7}{10}$

(b) $\frac{3}{10} \times 3 =$ ____

(c) $\frac{7}{10} \times \frac{9}{10} =$ ____

(d) $\frac{13}{10} \times \frac{3}{100} =$ ____

(e) $\frac{11}{10} \times$ ____ $= \frac{121}{100}$

(f) ____ $\times \frac{7}{10} = \frac{21}{100}$

(g) $\frac{1}{10} \times$ ____ $= \frac{19}{1000}$

(h) $\frac{31}{10} \times$ ____ $= \frac{124}{1000}$

The same examples written in decimal shorthand:

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
7	.3	.7	1.3	1.1		.1	3.1
x .1	x 3	x .9	x .03	x ___	x .7	x ___	x ___
				1.21	.21	.019	.124

$\frac{25}{10} \times \frac{18}{10} =$ ____

2.5
x 1.8
_____

(i)

$\frac{173}{100} \times \frac{31}{10} =$ ____

1.73
x 3.1
_____

(j)

$3\frac{7}{10} \times \frac{5}{10} =$ ____

3.7
x ___
_____

(k)

$197 \times \frac{9}{100} =$ ____

x ___
_____

(l)

.5
x 3
_____

(m)

2.5
x .6
_____

(n)

12.5
x .12
_____

(o)

6.25
x .24
_____

(p)

I. Undoing Multiplication . . . or, Dividing

1.	2.	3.	4.	5.	6.
5 x 3 — 15 5 3)15	x .4 — 2.8 .4)2.8	x 5 — 1.35 5)1.35	x .7 — 1.26 .7)1.26	x .08 — .112 .08).112	x .6 — .054 .6).054

7.	8.	9.	10.	11.	12.
2.16 x 3 — 6.48 3)6.48	x .5 — .870 .5).870	x .02 — .592 .02).592	x .6 — 51.0 .6)51	x .4 — .220 .4).22	x .8 — 6.00 .8)6

13.	14.	15.	16.	17.	18.
x 4 — 1.00 4)1.00	x 2 — 1.0 2)1	x 8 — 1.000 8)1	x 4 — 4)3	x 8 — 8)3	x 8 — 8)7

19.	20.	21.	22.	23.
x 1.7 — 175 5 — 4.25 2 1.7)4.25 34 ----- -----	x .43 .43)159.1	x .18 .18)81	x .50 .50)31	x .014 .014)7

II.

	Purchase price shown on the cash register	Money given to the clerk	Amount of change
a.	$ 0 0 0 7 5	$1.00	$.25
b.	$ 0 0 1 1 0	$5.00	
c.	$ 0 1 2 5 0	$20.00	
d.	$ 0 0 5 0 7	$10.00	
e.	$	$5.00	$3.83

	Purchase price shown on the cash register	Money given to the clerk	Amount of change
f.	$	$60.00	$1.82
g.	$ 0 2 7 4 6	$100.00	
h.	$ 0 3 0 8 1		$6.19
i.	$ 0 0 8 7 9	$10.00	
j.	$	$25.00	$3.09
k.	$ 4 8 6 2 9	$1,000.00	
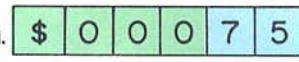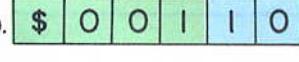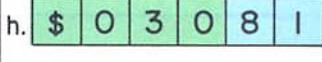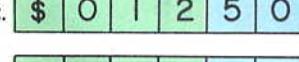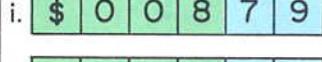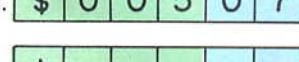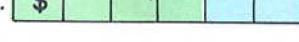
You write | Machine writes

I. Another setting on Alec's machine gave these results.

Find the pattern and complete the blue section.

You write:

$\frac{1}{4}$ $\frac{1}{3}$ $\frac{25}{100}$ $\frac{2}{10}$ 17% .5 18 50% 100% $\frac{3}{8}$ $\frac{3}{10}$ $\frac{29}{100}$ .18 $\frac{23}{100}$ $\frac{3}{4}-\frac{1}{2}$ 25% .09 .25 $\frac{5}{8}-\frac{1}{2}$ 2.1 $\frac{25}{10}$ 5% $\frac{1}{5}$

Machine writes:

$< \frac{1}{4}$	$\frac{1}{4}$	$> \frac{1}{4}$
17%	$\frac{1}{4}$	.5
$\frac{23}{100}$	$\frac{3}{4}-\frac{1}{2}$	18
$\frac{1}{5}$	25% $\frac{1}{3}$	

II. To complete the record, you need to know what the shorthand "%" means. The percent sign is just another sign for "hundredths." Maybe it had a history like this:

$$\frac{25}{100} \rightarrow \frac{25}{o/o} \rightarrow \frac{25}{\%} \rightarrow 25\%$$

To say "25%" amounts to saying "25/100." So, we can say:

$$\frac{25}{100} = \frac{1}{4} = .25 = 25\% = \frac{2}{8} = \frac{5}{20} \text{ etc.}$$

$$\frac{50}{100} = \frac{}{2} = .50 = .5 = \underline{}\% = \frac{}{30}$$

$$\frac{1}{5} = \frac{}{10} = \frac{}{100} = \underline{}\% = .\underline{} = \frac{}{15}$$

$$\frac{3}{4} = \frac{}{40} = \frac{}{100} = \underline{}\% = .\underline{} = \frac{}{16}$$

III. Complete the following, using:

= is equal to
< is less than
> is greater than

$\frac{1}{2}$ ☐= 50% $\frac{1}{4}$ ☐< 30%   10% ☐ $\frac{10}{100}$

$\frac{1}{3}$ ☐ 25% $\frac{2}{5}$ ☐ 40%   75% ☐ .8

90% ☐ $\frac{90}{100}$ 100% ☐ $\frac{100}{100}$ 100% ☐ 1

125% ☐ $1\frac{1}{4}$ 150% ☐ $1\frac{3}{4}$ 200% ☐ $1\frac{1}{2}$

23% ☐ $\frac{1}{4}$ $\frac{3}{4}$ ☐ 72%   1% ☐ .1

15% ☐ $\frac{3}{20}$ $\frac{1}{4}$ ☐ 30%   1% ☐ $\frac{1}{2}$

IV. Here are some common uses of this shorthand:

25% off on all items.

We pay 3% on all savings deposits.

We won 50% of our games.

35% Dacron, 65% cotton.

100% Beef Frankfurters.

Pay only 75% of marked price.

What do these sentences tell you?

V. Below, write sentences that include the following expressions:

50%   10%   4%   125%

103

I. Complete the following sentences.

1. 50% of 38 = _____   6. 25% of _____ = 7   11. _____% of 4 = 3

2. 8 = 10% of _____   7. 2 = 20% of _____   12. 1 = 5% of _____

3. _____% x 36 = 9   8. _____ = 40% x 40   13. 75% x 80 = _____

4. 15 = 15% x _____   9. 150% x 20 = _____   14. 12 = _____% x 8

5. _____ = 15% of 10   10. 50 = _____% x 25   15. _____% x 20 = 4

II. Each box below is to contain a "part" of the amount on the left. The relationship of that part to the whole amount is indicated at the top of the column.

		50%	25%	10%	75%	5%	150%
a.	1 hour	min.	15 min.	min.	min.	min.	90 min.
b.	1 yard and 4 inches	in.	in.	4 in.	in.	in.	in.
c.	5 dollars	$	$	$	$	$	$

		10%	_____%	15%	110%	130%	_____%
d.	$25.00	$	$5.00	$	$	$	$62.50
e.	_____ minutes	sec.	24 sec.	sec.	sec.	sec.	5 min.
f.	_____ feet	1 ft.	ft.	in.	ft.	ft.	ft.

III. Write an appropriate story for each of the three headlines given below.

a. | 40% x 25 = _____ |

b. | 10% x _____ = 7 |

c. | _____% of 9 = 2.25 |

104

I. Rearrange the numbers given in the green region in the order indicated by the signs.

Remember: < is less than
           > is greater than
           = is equal to

a.	$\frac{1}{2}$	75%	.4	$\frac{8}{10}$	.49	41%	.4 < 41% < .49 <	<	<
b.	.7	125%	1	$\frac{16}{10}$	85%	1.5	>	>	>	>	>
c.	$\frac{1}{2}$	.03	3	.3	30	$\frac{1}{4}$	<	<	<	<	<
d.	1.48	$\frac{9}{10}$	99%	150%	$1\frac{1}{100}$	$1\frac{1}{2}$	150% = $1\frac{1}{2}$ >	>	>	>
e.	$\frac{3}{10}$	20%	$\frac{1}{4}$	$\frac{3}{5}$	$\frac{1}{5}$	60%	=	>	>	>	=
f.	1.00	$\frac{6}{3}$	200%	125%	1.9	$\frac{10}{8}$	<	=	<	<	=
g.	39%	$\frac{9}{20}$	$\frac{3}{100}$	.49	2%	$\frac{2}{5}$	<	<	<	<	<
h.	$\frac{1}{8}$	1.0	6	$\frac{8}{2}$	$\frac{4}{36}$	12%	>	>	>	>	>
i.	$\frac{7}{20}$	.06	.35	6%	$\frac{3}{50}$	35.0%	=	=	>	=	=
j.	.75	1000%	50%	10	$\frac{15}{20}$	$\frac{11}{22}$	=	<	=	<	=
k.	$\frac{11}{100}$	$\frac{7}{50}$	13%	$\frac{3}{25}$	.1	9%	<	<	<	<	<
l.	$\frac{1}{20}$	$\frac{1}{100}$	$\frac{5}{2}$	5%	3	250%	>	=	>	=	>

II. Using < , > , or = , make three statements about the number given in the green region.

A. 25% x 7	> 1	< 2	> 1.5
B. 10% x 83	8.00	80.0	800
C. 1% x 97	10	1	.7
D. .10 x 25	.20	2.0	20
E. $\frac{10}{11}+\frac{19}{20}$	.5	2.5	1
F. .20 x 37	.6	7	18
G. $\frac{1}{3}+\frac{1}{2}$	$\frac{1}{6}$	1	5
H. $\frac{7}{10}-\frac{30}{100}$	$\frac{1}{7}$	.4	2.1
I. 7 ÷ 2	34	.4	4
J. .7 ÷ 2	.3	5	1.4

K. 3 x $\frac{11}{10}$	2	7	10
L. 3 x .2	$\frac{6}{1}$	$\frac{6}{10}$	$\frac{6}{100}$
M. .3 x .2	$\frac{6}{1}$	$\frac{6}{10}$	$\frac{6}{100}$
N. .5 x .7	35	3.5	.35
O. 8 x .4	32	3.2	.32
P. 1.2 x 1.2	1	2	14.4
Q. 1.0 x 9	.90	.09	9.0
R. .3 x 25	7.5	$\frac{3}{8}$	75%
S. 5 x .5	1%	10%	100%
T. .8 x .5	4.0	.04	.4

Watch the Signs! (Use scratch paper if you need to.)

I.

+	17	46	
24	41		
70			
52	69		140
			125

II.

X		4	9
	42		54
3		12	
8	35		

III.

+	.5	2.5	5	.25
8	8.5			
1.8				
.18				
.8				

IV.

+	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{3}{4}$
$\frac{1}{2}$	1			
$\frac{1}{3}$				
$\frac{2}{3}$				

V.

X	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{3}{4}$
$\frac{1}{2}$	$\frac{1}{4}$			
$\frac{1}{3}$				
$\frac{2}{3}$				

VI.

+	$3\frac{1}{2}$	$1\frac{5}{6}$	$4\frac{3}{8}$	$\frac{3}{10}$
$4\frac{1}{3}$		$6\frac{1}{6}$		
$2\frac{1}{2}$				
$\frac{2}{10}$				

VII.

−	50		57
	22		
		71	44
		35	
37		47	20

VIII.

X	3	.2	.08
1.2			.096
.5			
14			
.9			

IX.

X	10	.19	4.1
3	30		
.2			
.005			
.7			

X. Mr. Boles sold floor tile. Some of the tiles were 1 foot square — 12 inches by 12 inches — and were sold only in packs of 1 dozen each. To save time, Mr. Boles made a chart so that he could quickly answer questions as soon as he knew the length and width of the floor to be covered with tile. Fill in the blanks in the part of his chart shown below.

(In the chart below, all rooms are more than 4 feet in length and in width.) All lengths and widths are given to the nearest foot.

	Length of the room	Width of the room	No. of tiles needed	No. of packages needed	No. of tiles not used
a.	7 ft.	5 ft.	35	3	1
b.	7 ft.	6 ft.			
c.	7 ft.	7 ft.			
d.	7 ft.	8 ft.			
e.	7 ft.	9 ft.			

	Length of the room	Width of the room	No. of tiles needed	No. of packages needed	No. of tiles not used
f.			55		
g.				7	9
h.		15	225		
i.	14			19	4
j.		7			5

GOAL | 10 | | | | | 50 | 40 | | | | GOAL
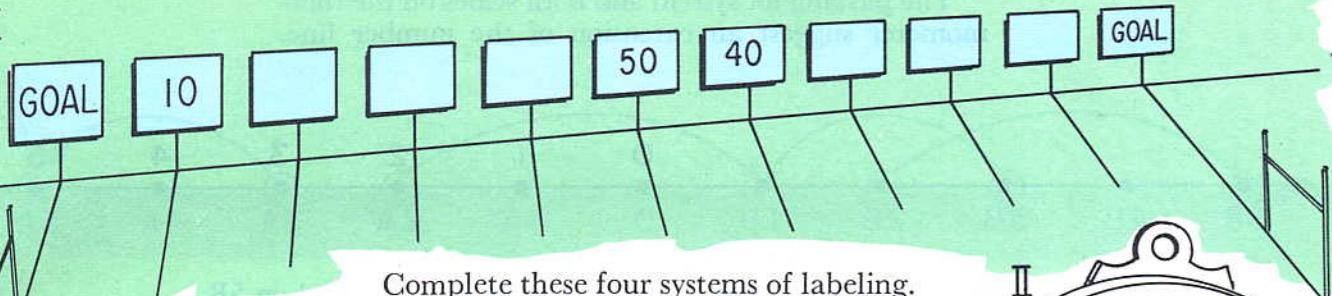
Complete these four systems of labeling.

II

C | F

III

Jack | Jill

School | Art

3rd N

Dick | Library
Mary

2nd N

Ed | Phil

Betty

1st N

Tom

Hardware Store | Monument | Bank

MAIN W | City Hall | MAIN E

CENTER N

1st S

Jane | Bill

Theatre

Helen

Bob

1st W | CENTER S | 1st E | 2nd E | 3rd E

90 — | — 200
80 —

10 — | — 60
0 — | — 40
10 — | — 20

— 0

— 20

— 80

IV

PARKING

2B | 1B | ENTER | | 2A | 3A | | 5
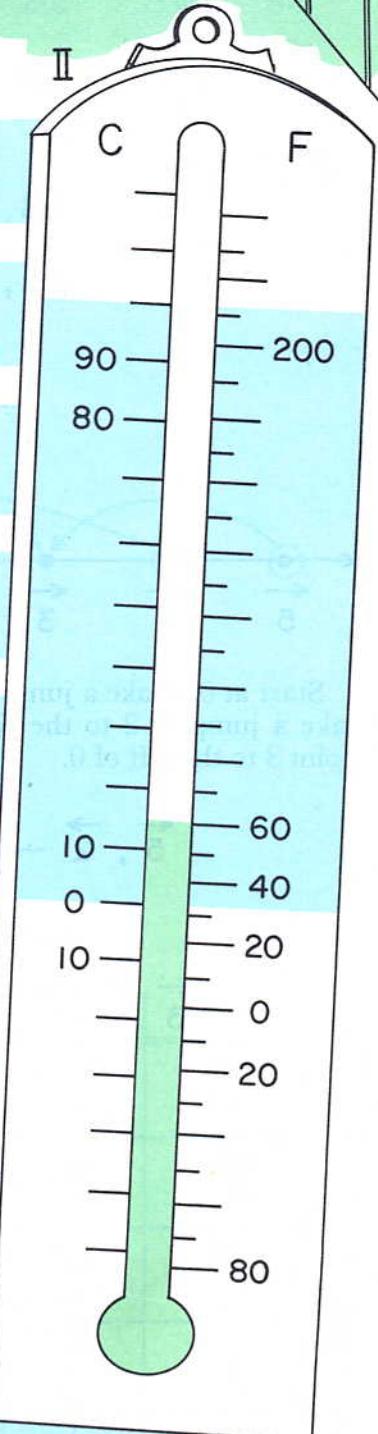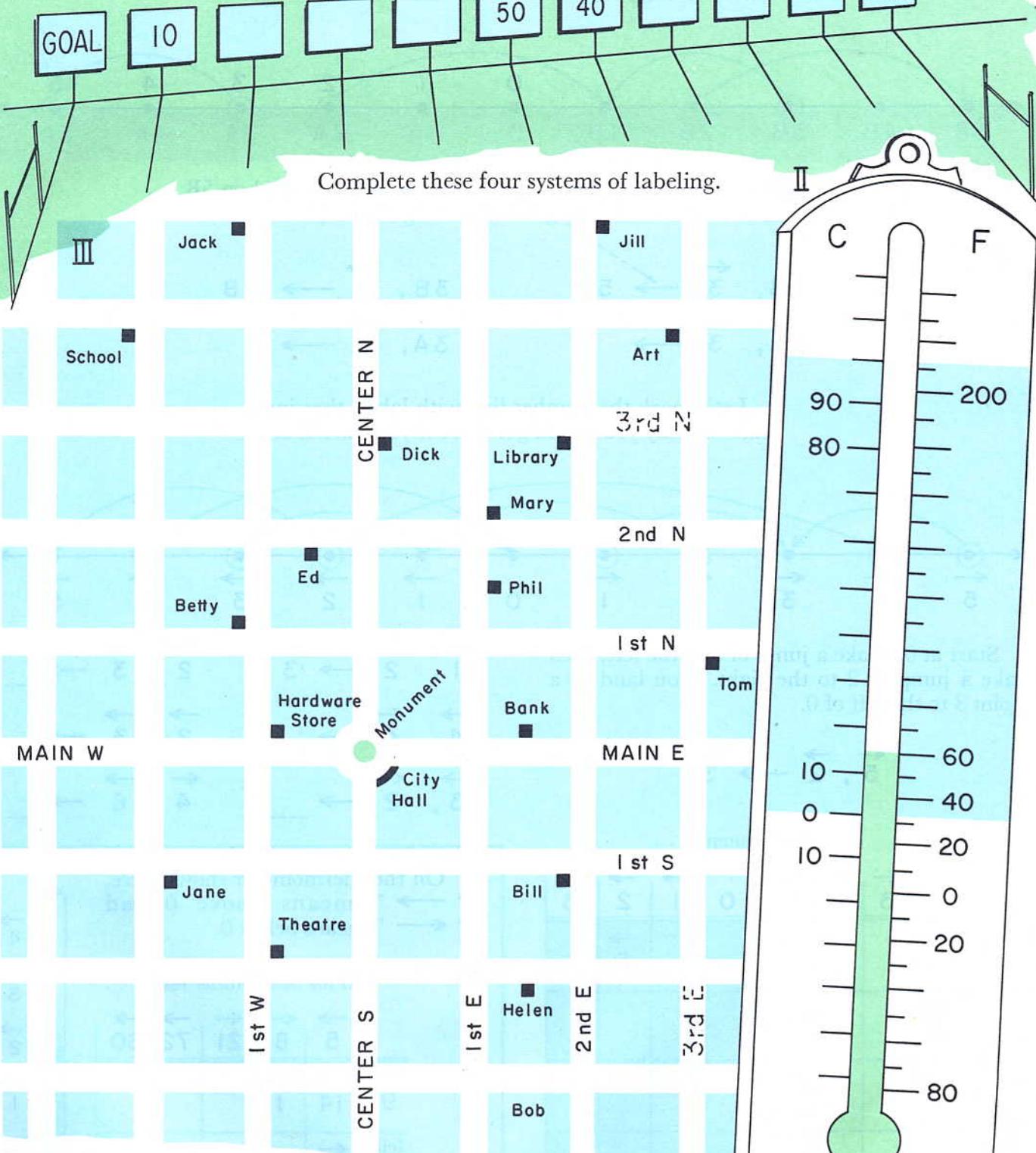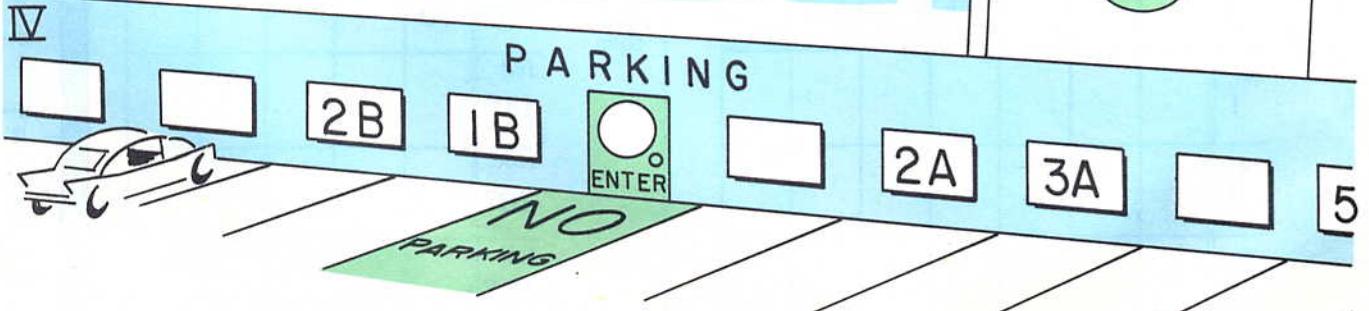
NO PARKING

The parking lot system and both scales on the ther-
mometer suggest an extension of the number line.

| 5B | 4B | 3B | 2B | 1B | 0 | 1A | 2A | 3A | 4A | 5A |
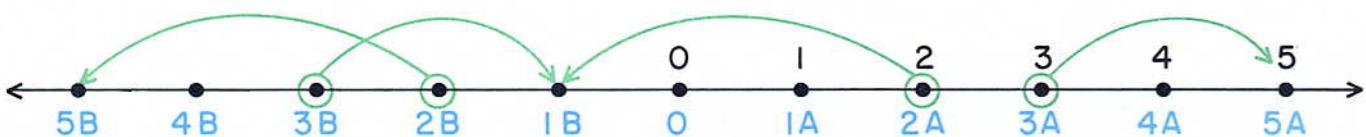
If you start at 2B and jump 3 spaces to the left, you land on 5B.
(Read that larger arrow as "goes to.")

2B, ←3 ⟶ 5B   3B, →2 ⟶ 1B

2A, ←3 ⟶   3A, →2 ⟶

Let's mark the number line with labels that indi-
cate the way you would get there if you started at 0.

| ←5 | ←4 | ←3 | ←2 | ←1 | 0 | →1 | →2 | →3 | →4 | →5 |
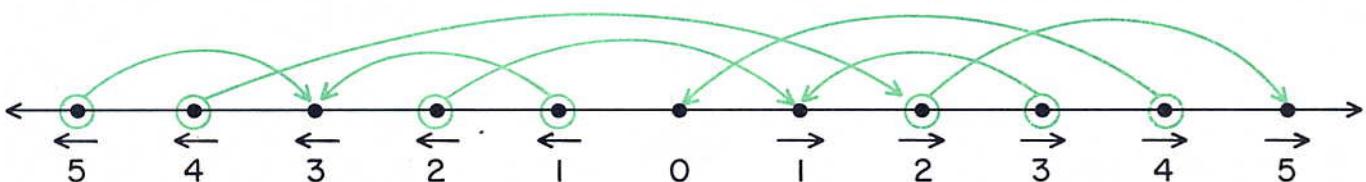
Start at 0. Take a jump of 5 to the left, then
take a jump of 2 to the right. You land at a
point 3 to the left of 0.

←5, →2 ⟶ ←3

←1, ←2 ⟶ ←3   ←2, →3 ⟶ ____

→4, ←4 ⟶ ____   →2, →3 ⟶ ____

→3, ←2 ⟶ ____   ←4, →6 ⟶ ____

First a jump of . . .

On the thermometer shown here,
" ⟶ " means above 0 and
" ⟵ " means below 0.

If the thermometer read . . .

	→5	←8	←21	→72	→60
rose 9°	→14	→1			
fell 15°	←10				
fell 11°					
rose 28°					
fell 16°					

and then . . .

108
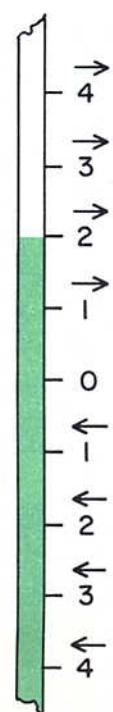
In the "headlines" below, arrows are used to indicate directions. They can be used for many kinds of directions. For example, if → means *forward* then ← will mean *backward*. We say that the direction → is the opposite of the direction ←, and that ← is the opposite of →. *(Fill in the missing directions.)*

If → means . . .	Up	Profit	_____	_____	Found	_____
then ← means . . .	Down	_____	West	Below	_____	_____

Write a story for each headline, and write a headline for each story.

a | $\overrightarrow{17}$, $\overleftarrow{25}$ → ____ | *Mr. Jones drove 17 miles north from his home on Route 7. Then he turned around and drove 25 miles south. Where was he then?*

b | $\overleftarrow{8}$, ____ → $\overrightarrow{21}$

c | $\overleftarrow{1.25}$, ____ → $\overleftarrow{5}$

d | ____ , $\overleftarrow{4}$ → $\overleftarrow{1\frac{1}{2}}$

e | $\overrightarrow{5}$, $\overleftarrow{13}$, $\overrightarrow{15}$ → ___

f | Bill started down the basement stairs. After going down 8 stairs, he changed his mind. Then he went up 17 stairs.

g | Harry lost $1\frac{1}{2}$ pounds one week and twice as much the next. How had his weight changed?

h | The skin diver went down 63 feet. After he came up 25 feet, he saw a beautiful piece of coral. How far was the coral beneath the surface?

i | Mr. Swanson found he had gone 5 miles in the wrong direction. "Now I'm 32 miles from home." How far would he have had to go if he had not made a mistake?

j | Bill walked 175 feet into a cave. Then he turned around and walked 210 feet. Where was he?

k | The water had cooled 75 degrees Fahrenheit after it had boiled. This happened at the seashore last summer.

"Directed numbers" is one of the names that has been given to those numbers that can be used to label points on the number line on both sides of 0. Where do you think that name came from?

Others call them "signed numbers." Why might that be an appropriate name?

Most mathematicians do not use arrows such as we have used . . . $\overrightarrow{3}$ or $\overleftarrow{4}$. They prefer to indicate a direction from 0 in the following way:

+3 . . . read "positive three," and
−4 . . . read "negative four."

We shall use these customary symbols.

What do you suppose we could mean by
ADDITION of directed numbers?

Here is a picture using the idea of doing addition with a number ladder:
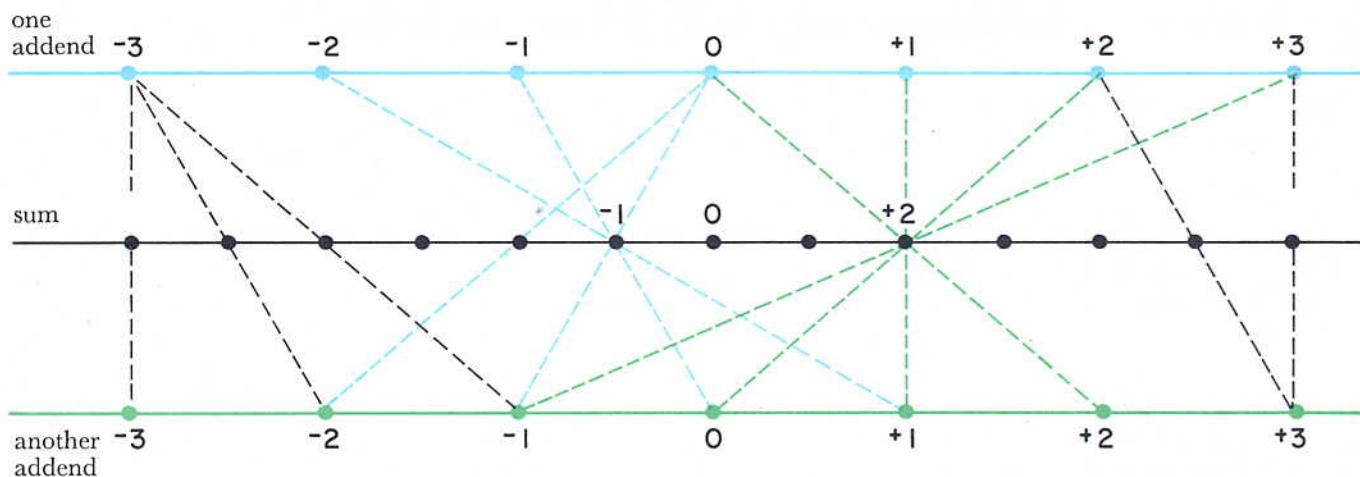
Can you finish labeling the middle line?

Complete the following sentences.

1. $(\,^+1\,) + (\,^+1\,) = ^+2$
2. $(\,0\,) + (\,^+2\,) =$
3. $(\,^+2\,) + (\,0\,) =$
4. $(\,^+3\,) + (\,^-1\,) =$
5. $(\,^-2\,) + (\,^+1\,) =$
6. $(\,^-1\,) + (\,0\,) =$
7. $(\,0\,) + (\,^-1\,) =$
8. $(\,^+1\,) + (\,^-2\,) =$
9. $(\,^-3\,) + (\,^-1\,) =$
10. $(\,^-3\,) + (\,^-2\,) =$
11. $(\,^-3\,) + (\,^-3\,) =$

12. $^+1 = (\,^-1\,) + (\quad)$
13. $^-2 = (\quad) + (\,^+1\,)$
14. $(\,^-3\,) + (\quad) = 0$
15. $\quad = (\,^-1\,) + (\,^+1\,)$
16. $(\quad) + (\,^+2\,) = ^+5$
17. $(\,^-2\,) + (\quad) = ^+2$
18. $^-5 = (\,^-1\,) + (\quad)$
19. $^-15 = (\,^-7\,) + (\quad)$
20. $(\,^-6\,) + (\,^+15\,) =$
21. $^+6 = (\quad) + (\,^-9\,)$
22. $(\quad) + (\,^+3\,) = ^-7$

23. $(\,^+3\,) + (\,^-2\,) = (\quad) + (\,^+3\,)$
24. $(\,^-2\,) + (\,^-3\,) + (\,^+4\,) =$
25. $(\,^+4\,) + (\,^-2\,) + (\,^-3\,) =$
26. $(\,^+2\,) + (\quad) = (\,^-3\,) + (\,^-2\,)$
27. $(\,^+9\,) + (\quad) = (\,^-5\,) + (\,^+9\,)$
28. $(\,^-4\,) + (\,^-4\,) + (\,^-4\,) =$
29. $(\,^+5\,) + (\,^+5\,) + (\,^+5\,) =$
30. $(\,^-17\,) + (\,^-8\,) + (\,^+17\,) =$
31. $(\,^-25\,) + (\,^+9\,) + (\,^+16\,) =$
32. $^+\quad = (\,^-\quad) + (\,^+\quad) + (\,^-\quad)$
33. $(\,^-\quad) + (\,^+\quad) + (\,^-\quad) = ^-$

110

Think of a Number

"I once played a game that went like this," Harry said.

"Think of a number. Don't tell me, but write it down so you won't forget.

"Add 3. Double your result. Subtract 4. Divide by 2. Subtract the number you thought of first. If your answer isn't 1, then you've made a mistake!"

Most of the class was amazed — except one or two who remembered the trick. The class made a chart of what had happened.

(Please complete the record.)

	Al	Bob	Don	Meg	Art	Sam	Alice	Jim	Nancy	Paul	Helen
Think of a number.	5	7	0	3	25					158	589
Add 3.	8					34					
Double the result.	16						44				
Subtract 4.	12							56			
Divide by 2.	6								106		
Subtract the number you thought of first.	1										

The picture below suggests the way this trick works.

A convenient shorthand

Think of a number.		THINK OF A NUMBER. Imagine that you put that many counters in a bag. Don't tell anyone how many.	
Add 3.		ADD THREE. Put 3 counters by the bag. Now you have a bag and 3 counters more.	$X + 3$
Double it.		DOUBLE IT. You need another bag like the first, and another 3 counters. That's 2 bags and 6 counters.	$2X + 6$
Subtract 4.		SUBTRACT FOUR. That leaves 2 bags and 2 counters.	$2X + 2$
Divide by 2.		DIVIDE BY TWO. Take away one bag and one counter, leaving one bag and one counter.	$X + 1$
Subtract the number you thought of first.	•	SUBTRACT THE NUMBER YOU THOUGHT OF FIRST. No matter how many counters you had in it. One left!	1

Remarks about the shorthand.

In the shorthand, we erase the top and bottom of the bag, and the remainder looks much like the letter *x*. Of course, any letter would do to hold a place for "the number you thought of first" — *n* or *y* — you choose it.

Instead of drawing counters, we used numbers to keep track of "how many."

Instead of drawing two bags, we shortened that to 2*x*. In other games we may wish to show 100 bags, and 100*x* is a short way of indicating that.

111

More about the Shorthand

We shall use x to hold a place for any number you think of. We could use any mark or letter, but on this page we shall use x.

In the spaces that follow, choose your answer from the expressions in the blue-tinted block at the top of the next column.

Choose the expression that best fits the description given. The first two blanks are filled in to show what is wanted.

$$x \quad x+1 \quad x+2 \quad x+3 \quad 2x$$
$$x-1 \quad x-2 \quad 3x \quad x^2 \quad x1$$

(x^2 is shorthand for x times x — "x squared.")

a. any number you may think of _x_

b. one more than the number you may have thought of _$x+1$_

c. one less than the number you thought of _____

d. two less than the number you thought of _____

e. three more than the number you thought of _____

f. three times the number you thought of _____

g. two more than the number you thought of _____

h. the number you thought of multiplied by two _____

i. the number you thought of multiplied by itself _____

j. twice the number you thought of _____

k. the number you thought of with one subtracted _____

l. the number you thought of with two added _____

m. three added to the number you thought of _____

n. three multiplied by the number you thought of _____

o. the square of the number you thought of _____

p. the number you thought of multiplied by one _____

"Think of another number," Harry broke in.

"Then follow these directions: first double it. Next, add 9. Then add the number you thought of first. Divide by 3. Double this result. Subtract 6 and tell me your answer. I'll tell you the number you thought of first."

	Shorthand	Al	Bob	Don	Meg	Art	Helen	Alice
Think of a number.	x	4	2					
Double it.	$2x$	8	4					
Add 9.	$2x+9$	17						263
Add the number you thought of first.	$3x+9$	21						
Divide by 3.	$x+3$	7					78	
Double this result.								
Subtract 6.		8			0	100		
The number you thought of is	x	4	2	1				

More Shorthand

Each series of expressions include some of the different ways of saying the same thing. The usually preferred form is printed on a green background.

A. two times three . . . three times two . . . 2 multiplied by 3 . . . 2×3 . . . $2 \cdot 3$ . . . $3 \cdot 2$. (The multiplication sign looks so much like the letter x we often use a dot in its place.)

B. the number you thought of times 4 . . . $x \cdot 4$ . . . $4 \cdot x$ . . . $4x$. (Where no confusion will result, we omit the dot. Why shouldn't we omit the dot in $2 \cdot 5$?)

C. twice the sum of five and nine . . . $2 \times (5 + 9)$ . . . $2 \cdot (5 + 9)$ . . . $2(5 + 9)$. (We can omit the dot here because no confusion results.)

D. the number you thought of . . . x . . . a . . . n . . . y . . . z . . . b . . . etc. (Any letter is as good as another. Be sure each time everyone understands what you mean.)

An agreement about the order of work to be done.

To prevent confusion, mathematicians agree that all indicated multiplications and divisions should be carried out first. Only then should indicated additions and subtractions be carried out. (If some different order is to be followed, parentheses are placed around the part of the expression to be dealt with first.)

Without this agreement, we could not tell which of the following was indicated:

Which one?

$$2 + 4 \cdot 3 = 6 \cdot 3 = 18$$
$$2 + 4 \cdot 3 = 2 + 12 = 14$$

Certainly 18 and 14 are very different results. But, because of our agreement about order of work, we know that the indicated multiplication should be carried out first.

If it were intended that the addition should be done first, that would have been indicated by parentheses. It would be:

$$(2 + 4)3 = 6 \cdot 3 = 18$$

With this agreement in mind, complete the mathematical sentences — a through l — in the column at the right.

a. $2 \cdot 3 + 1 =$ ____

b. $2 + 3 \cdot 4 =$ ____

c. $3 \cdot 2 - 1 =$ ____

d. $1 + 2 \cdot 3 =$ ____

e. $1 + 3 \cdot 2 =$ ____

f. $(2 + 3) \cdot 4 =$ ____

g. $2 \cdot (3 - 1) =$ ____

h. $4(3 + 2) =$ ____

i. $(3 - 1)5 =$ ____

j. $3(2 - 1) =$ ____

k. $(1 + 2)3 =$ ____

l. $(1 + 3)2 =$ ____

In the lines below, is the expression printed in green equivalent to the expression printed in blue?

Are they equivalent?

1. $1 + 2 \cdot 3$ . . . . $1 + 3 \cdot 2$ . . . . _____

2. $(1 + 2)3$ . . . . $(1 + 3)2$ . . . . _____

3. $2 \cdot 3 + 1$ . . . . $2(3 + 1)$ . . . . _____

4. $(2 + 3)4$ . . . . $2 \cdot 4 + 3 \cdot 4$ . . . . . . _____

5. $2(3 + 4)$ . . . . $(3 + 4)2$ . . . . _____

6. $3(2 + 4)$ . . . . $3 \cdot 2 + 3 \cdot 4$ . . . . . . _____

7. $2 + 3 \cdot 4 + 5$ . . . . $(2 + 3) \cdot (4 + 5)$ . . _____

8. $3x + 2$ . . . . $3(x + 2)$ . . . . _____

9. $2(x + 7)$ . . . . : . . $2x + 14$ . . . . _____
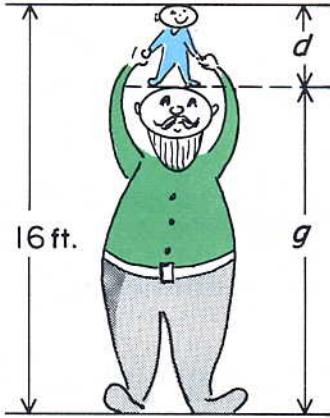
I. A dwarf stands on a giant's head. Together, they are 16 feet tall. What is the height of the dwarf? What is the height of the giant?

Obviously, we don't have enough information to answer the questions. However, we can fill in the chart below:

If the height of the giant and dwarf together is 16 feet

	a.	b.	c.	d.	e.	f.	g.
and the height of the dwarf in feet is	3	2	1	$2\frac{1}{2}$	$1\frac{1}{3}$	$\frac{7}{8}$	$\frac{11}{12}$
then the height of the giant in feet is							

In the shorthand of algebra, we can write:

(a)
If .... $d + g = 16$
and if .... $d = 3$
then .... $g = 13$.

(b)
If .... $d + g = 16$
and .... $d = 2$
then .... $g = \_\_\_$.

(c)
If .... $d + g = 16$
and .... $d = 1$
then .... $g = \_\_\_$.

(d)
$d + g = 16$
$d = 2\frac{1}{2}$
$g = \_\_\_$

II. Bill has a board that is $7\frac{1}{2}$ feet long. He also has a shorter board. How far would the two boards reach if they are placed end to end? How long is the short board?

Again, we haven't enough information but we can fill out the chart that lists some of the possibilities.
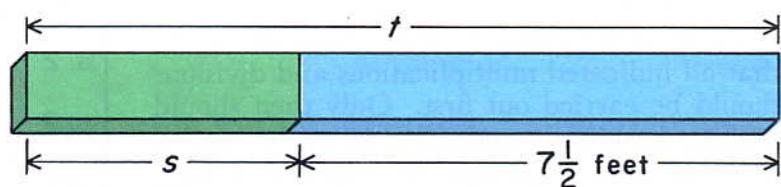
If the long board is $7\frac{1}{2}$ feet long

	a.	b.	c.	d.	e.
and the length in feet when placed end-to-end is		12		14	
then the length in feet of the short board is	$2\frac{1}{2}$		1		$3\frac{5}{8}$

In the shorthand of algebra, we can write:

(a)
If .... $s + 7\frac{1}{2} = t$
and .... $s = 2\frac{1}{2}$
then .... $t = 10$.

(b)
If .... $s + 7\frac{1}{2} = t$
and .... $t = 12$
then .... $s = \_\_\_$.

(c)
If .... $s + 7\frac{1}{2} = t$
and .... $s = 1$
then .... $t = \_\_\_$.

(d)
$s + 7\frac{1}{2} = t$
$t = 14$
$s = \_\_\_$

III. Two dwarfs of the same size stand, one on top of the other, on the head of a giant. The top of the head of the higher dwarf is 27 feet from the ground. Without more information, we cannot know the height of the dwarfs or the giant. But —

If a giant and 2 dwarfs together are 27 feet tall

	a.	b.	c.	d.	e.
and if the giant's height in feet is	20	24	$21\frac{1}{2}$	$22\frac{1}{3}$	$25\frac{1}{4}$
then each dwarf's height in feet is	$3\frac{1}{2}$				

(a)
If .... $g + 2d = 27$
and .... $g = 20$
then .... $2d = \_\_\_$
and .... $d = \_\_\_$

(b)
$g + 2d = 27$
$g = 24$
$2d = \_\_\_$
$d = \_\_\_$

(c)
$g + 2d = 27$
$g = 21\frac{1}{2}$
$2d = \_\_\_$
$d = \_\_\_$

114

I.

Mary folded a long strip of paper and colored the region on one side of the fold green — on the other side, blue. She measured the length of the strip before it was folded — length of green plus length of blue. After it was folded, she measured the green part that was not covered by the blue.

She did this with many such strips and kept a record. Can you find the lengths of the blue and green regions in each example?
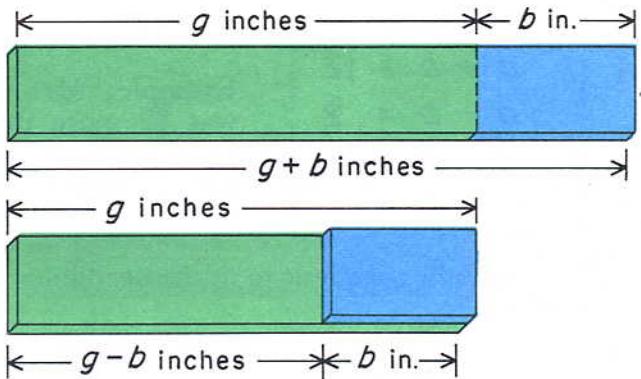
(All lengths are in inches.)

		a.	b.	c.	d.	e.	f.	g.	h.
Length of green and blue together	$(g+b)$	13	20	4	$15\frac{1}{2}$	$9\frac{1}{4}$			
Length of green not covered by blue	$(g-b)$	7	10	2	$4\frac{1}{2}$	$4\frac{3}{4}$		4	12
Length of green	(g)	10	15				18		21
Length of blue	(b)	3					7	$1\frac{1}{2}$	

(a)
If.... $g + b = 13$
and.. $g - b = 7$
then...... $g =$ ____
and...... $b =$ ____

(b)
If.... $g + b = 20$
and.. $g - b = 10$
then..... $g =$ ____
and...... $b =$ ____

(f)
If....... $g = 18$
and..... $b = 7$
then.. $g + b =$ ____
and.. $g - b =$ ____

(e)
If....... $g = 21$
and.. $g - b = 12$
then...... $b =$ ___
and.. $g + b =$ ___
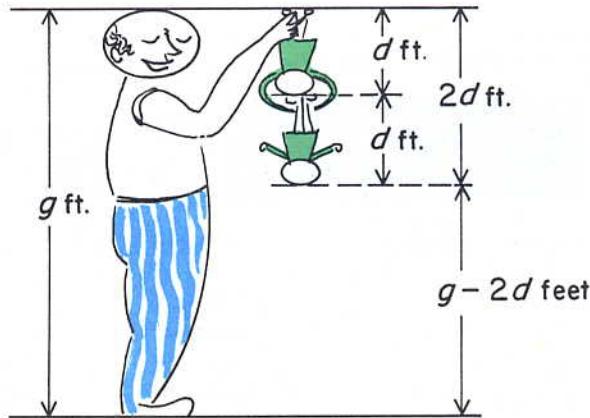
II.

A giant touches the ceiling with his head. He holds a dwarf's feet against the ceiling. This dwarf holds the feet of another dwarf.

The chart below indicates several possibilities. Of course, giants and dwarfs can be any height we wish to imagine. (All heights are in feet.)

		A.	B.	C.	D.	E.	F.	G.	H.
Height in feet of giant	(g)	20	100	18		$33\frac{3}{4}$		$50\frac{1}{2}$	75
Height in feet of two dwarfs	$(2d)$	4	3	1			9		
Height of one dwarf	(d)	2	$1\frac{1}{2}$		1	$1\frac{7}{8}$			$\frac{1}{4}$
Number of feet from bottom of dwarf's head to floor	$(g-2d)$	16			22		38	$44\frac{1}{2}$	

(B)
If........ $g = 100$
and..... $2d = 3$
then..... $d =$ ____
and.. $g - 2d =$ ____

(D)
If........ $d = 1$
and.. $g - 2d = 22$
then.... $2d =$ ____
and...... $g =$ ____

(F)
If....... $2d = 9$
and.. $g - 2d = 38$
then...... $g =$ ____
and...... $d =$ ____

(G)
If........ $g = 50\frac{1}{2}$
and.. $g - 2d = 44\frac{1}{2}$
then..... $2d =$ ____
and....... $d =$ ____

115

I.
$$a + b = 12$$
$$a - b = 2$$
$$a = \text{\_\_\_\_}$$
$$b = \text{\_\_\_\_}$$

Example: Mrs. Clark made two purchases. Her first purchase (a) was $2 more than the second ($b$). Altogether, she spent $12 ($a + b$ dollars). Therefore, the amount of her first purchase was _____; the second was _____.

Make up another example to fit the headline: _____

_____

_____

II.
$$x + 2y = 130$$
$$y = \text{\_\_\_\_}$$
$$2y = \text{\_\_\_\_}$$
$$x = \text{\_\_\_\_}$$

Example: Home — x mi. — Grandmother's — y mi. — Uncle Ed's — $2y$ mi.
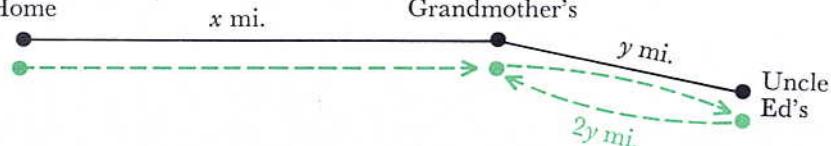
After leaving home, we drove to Grandmother's; then to Uncle Ed's; then back to Grandmother's. Altogether, we drove 130 miles. It is 20 miles from Grandmother's to Uncle Ed's. From Grandmother's, we shall have to drive_____miles to get home.

Make up another example to fit the headline: _____

_____

_____

III.
$$2r + t = \text{\_\_\_\_}$$
$$t = 24$$
$$2r = \text{\_\_\_\_}$$
$$r = \text{\_\_\_\_}$$

Mr. Brown bought 144 square tiles. He had two floors the same size to cover. When he finished, there were 2 dozen tiles left over. He used _____ tiles for each floor.

[diagram: r tiles | r tiles | t tiles left over]
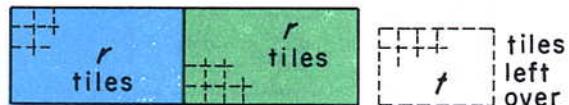
Make up another example to fit the headline: _____

_____

_____

IV.
$$x - y = 60$$
$$x + 2y = 75$$
$$x = \text{\_\_\_\_}$$
$$2y = \text{\_\_\_\_}$$
$$y = \text{\_\_\_\_}$$

Jack and Bill each weighed the same when school started in the fall. Jack was sick for several weeks and lost some weight, and by Thanksgiving he weighed only 60 pounds. Bill gained twice as much as Jack lost, and he weighed 75 pounds. When school started, each must have weighed _____ pounds. Jack lost _____ pounds, and Bill gained _____ pounds.

Make up another example to fit the headline: _____

_____

I. An ant started crawling from one corner of a rectangularly shaped room. He crawled along the wall the width (w feet) of the room plus the length (l feet) of the room. It took exactly 360 1-foot square tiles to cover the floor. How far did the ant crawl?

We do not have enough information, but we can work out several of the possibilities.

Ant starts here
Ant winds up here
w ft.
l ft.
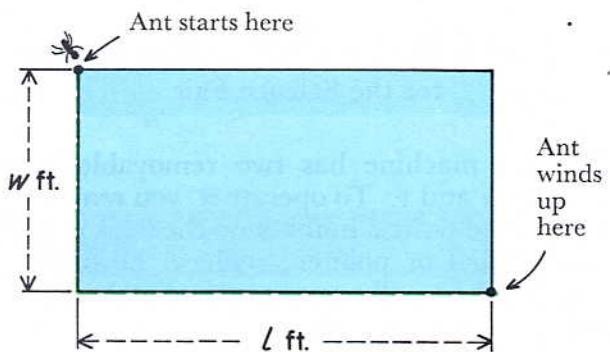
The area of floor is 360 square feet, or $l \cdot w = 360$.

	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
The length of the room.	30		60				72	90		
The width of the room.	12	15				3				
The distance the ant crawled.	42			38	46				49	53

In the shorthand of algebra, we can write:

(1)
If.... $l \cdot w$ = 360
and...... l = 30
then...... w = ____
and.. $l + w$ = ____ .

(4)
If.... $l \cdot w$ = 360
and.. $l + w$ = 38
then...... l = ____
and....... w = ____ .

(6)
If..... $l \cdot w$ = 360
and....... w = 3
then...... l = ____
and.. $l + w$ = ____ .

(9)
If... $l \cdot w$ = 360
and.. $l + w$ = 49
then...... l = ____
and....... w = ____

Ant starts and ends here
w ft.
l ft.

II. Now suppose that the ant crawls all the way around the room — back to the corner from which he started. All we know about the floor of the room is that it is rectangular in shape and covered with whole tiles that are 1-foot square.

The chart below considers some of the possibilities.

	1.	2.	3.	4.	5.	6.	7.	8.
The length of the room in feet (l).	10		20					
The width of the room in feet (w).	15	9				8		
The area of the room in square feet ($l \cdot w$).			340	225	77		221	143
The distance the ant crawls from one corner to opposite corner ($l + w$).		21						
Twice the distance the ant crawls $2(l + w)$.						34		
The total distance the ant crawls ($2l + 2w$).				60			60	

III. Suppose that the ant crawled around three of the walls and had 11 feet to go before reaching his starting place. The trip around the three walls had been 25 feet. The room must be _____ feet wide.

In the shorthand of algebra, we could write:

$w + l + w$ = 25
$2w + l$ = ____
l = ____
$2w$ = ____
w = ____

117

I. This machine has two removable blocks marked x and y. To operate it, you remove the blocks and write a number on the back of each; set the dial or pointer; replace the blocks — and numbers will appear immediately on each of the five blue screens.

Turn over and write number on back,

set dial at M,

replace blocks,
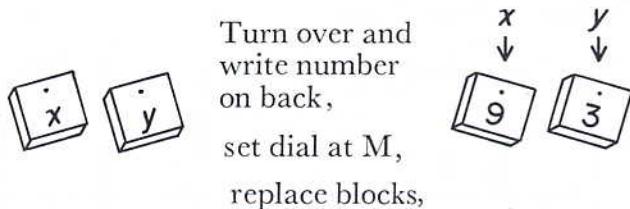
and the machine looks like this:
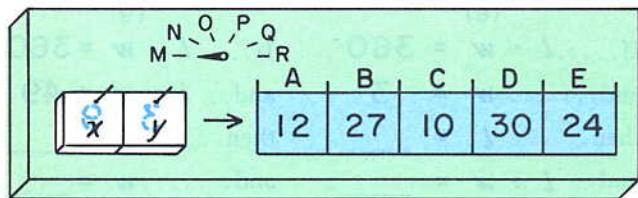
Keep changing the numbers written on the blocks and keep a record of the numbers the machine writes. Perhaps you can discover the rules. Here are some results with the pointer set at M:

x	y		A	B	C	D	E
9	3	→	12	27	10	30	24
4	6	→		24		27	20
11		→		22		25	26
		→	6	5			

Complete the chart above and the rules below:

Rule A: $x, y \rightarrow x + y$

Rule B: $x, y \rightarrow$

Rule C: $x, y \rightarrow x + y - 2$

Rule D: $x, y \rightarrow$

Rule E: $x, y \rightarrow$

II. When the pointer was moved into position N, the following results were noted. Tests began with a 9 written on the back of the x-block and a 3 on the back of the y-block.
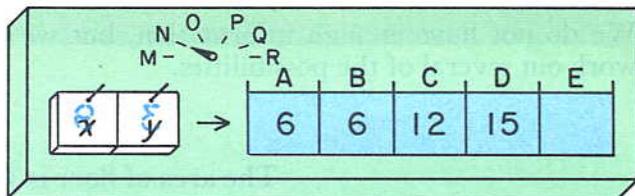
x	y		A	B	C	D	E
5	4	→	1	8		13	2
13	7	→				27	
		→	4	18			
		→		6		16	14

Rule A: $x, y \rightarrow$

Rule B: $x, y \rightarrow 2y$

Rule C: $x, y \rightarrow 2x - 2y$

Rule D: $x, y \rightarrow$

Rule E: $x, y \rightarrow 2(x - y)$

III. Here are results with the pointer set at O:

x	y		A	B	C	D	E
10	5	→	25	25	15	25	15
4	8	→	22		24		0
7	3	→		17	9	16	11
11		→	21	22	0	11	
	9	→	25	21		33	3

Rule A: $x, y \rightarrow x + y +$

Rule B: $x, y \rightarrow 2x +$

Rule C: $x, y \rightarrow$

Rule D: $x, y \rightarrow x +$

Rule E: $x, y \rightarrow 2x$

IV. Here are results with the pointer at P:

x	y		A	B	C	D	E
7	3	→	49	9	40	52	18
5	4	→		16	9	29	
8	1	→	1				
	6	→		36	64	106	
9		→			56	86	50

Rule A: $x , y \rightarrow x \cdot x$ or x^2

Rule B: $x , y \rightarrow$

Rule C: $x , y \rightarrow x^2 -$

Rule D: $x , y \rightarrow x^2 +$

Rule E: $x , y \rightarrow 2 \cdot y^2$ or $2y^2$

V. In the blue-tinted area are some expressions we have been using to complete the rules on these pages.

$$x^2 \qquad x + 3y \qquad x^2 - y^2$$
$$2x + 2y \qquad x + y - 2$$
$$2y \qquad y^2 \qquad 2(x + y)$$

Find sketches below that suggest these ideas.

(A)

This sketch suggests the idea of _ _ _ _ .

(B)

This suggests the idea of _ _ _ _ _ _ _ .

(C)

This suggests the idea of _ _ _ _ _ _ _ .

(D)
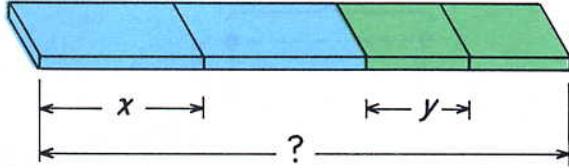
The sketch above suggests the idea of _ _ _ _ _ _ .

(E)
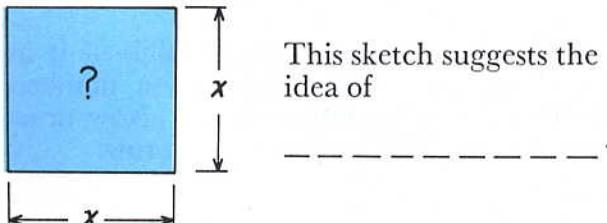
This sketch suggests the idea of _ _ _ _ _ _ _ _ _ .

(F)
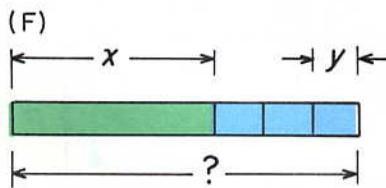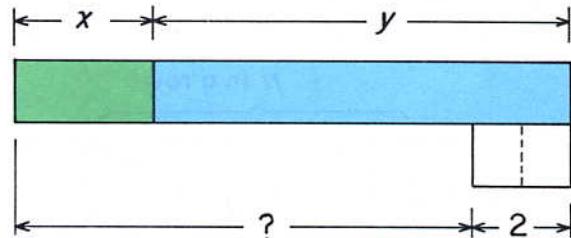
The above sketch suggests the idea of _ _ _ _ _ _ .

(G)
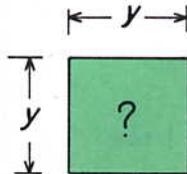
The sketch above suggests the idea _ _ _ _ _ .

(H)
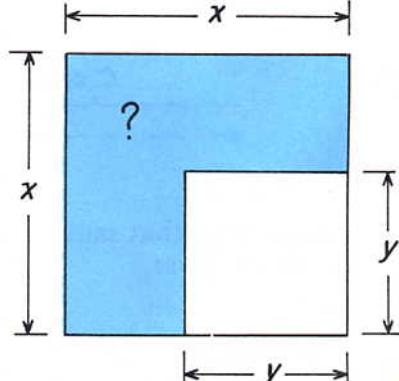
The blue part of the sketch above suggests the idea of _ _ _ _ _ _ _ _ _ .

We can use sketches to illustrate many of the ideas suggested by the shorthand of algebra.

In arithmetic, we use sketches or diagrams to see clearly what is going on. For example: there are four trees in each of two rows. A sketch might be:

4 in a row
} 2 rows
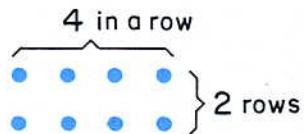
Because $2 \times 4 = 8$, there would be eight trees in all.

In some situations we don't have enough information to draw such a complete sketch. However, we can indicate at least as much as we do know. Here is an example:

Think of a number. Arrange that many counters in a single row. Only you know what number you thought of. Others can use a letter such as n to take the place of that number until they find out for sure what it is. Here is a sketch of the idea:
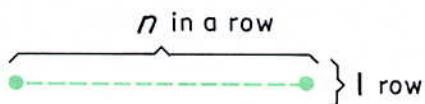
n in a row
} 1 row

This sketch will serve well unless the number you thought of was 0.

Think once again of a number. Arrange that many chairs in a row. We can say that you have arranged c chairs in a row, and sketch it as:
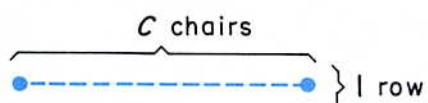
c chairs
} 1 row

Next, arrange that same number of chairs in each of three rows:
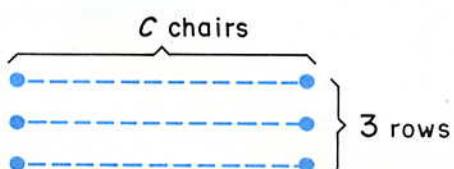
c chairs
} 3 rows

We can say that you arranged c chairs in each of 3 rows. So, there are $3 \times c$ chairs or $3 \cdot c$ or $3c$ chairs.

Think of another number. Draw that number of rows with 6 dots in each row. We shall use an a to remind us of the number you thought of.
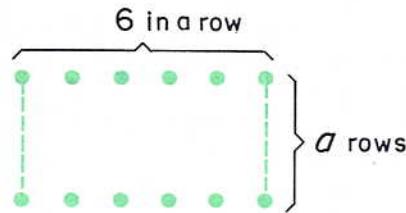
6 in a row
} a rows

You must have drawn $6 \times a$ or $6a$ dots.

Think of a number. (We shall use an s to remind us of that number until we know more about it.) Put that number of counters in a row. Add rows until you have that same number of rows.

s in a row
} s rows
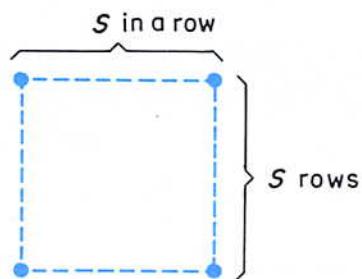
You must have, in all, $s \times s$ counters or s^2 counters; "s^2" is read as "s squared."

Think of a number. (We'll think of it as m.) And, once again, think of a number. (We'll use n to remind us of it.) Now draw n rows of dots with m dots in each row.
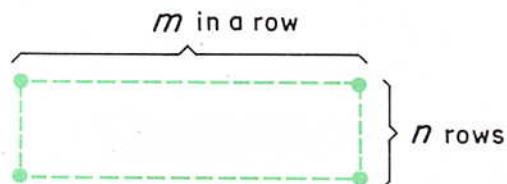
m in a row
} n rows

This sketch reminds us that you must have drawn $m \times n$ or mn dots.

120

Think of a number. Draw that many blue squares in a row.

x in a row

□------□ } 1 row
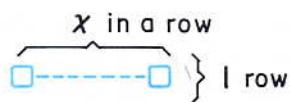
Think of another number. Extend the row with that number of green squares.

x y

□------□ □------□ } 1 row

$x + y$
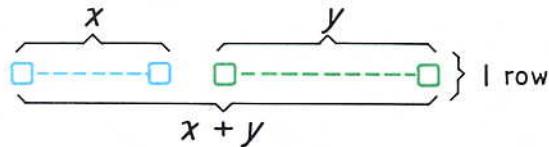
There are $x + y$ squares in a row. Now, draw 3 such rows.

x y

□------□ □------□
□------□ □------□ } 3
□------□ □------□
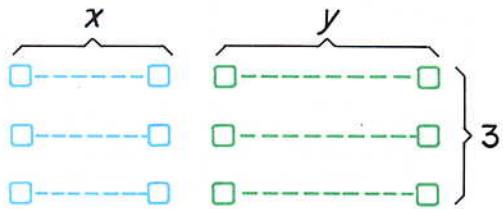
The sketch tells us that you must have drawn $3x + 3y$ squares. Or, we can say there must be $3(x + y)$ squares.

Think of a number and draw a row with that number of circles in it.

a

○------○
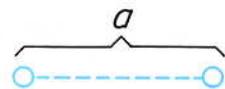
Add 3 circles to that row.

a 3

○------○ ○ ○ ○ } 1 row

$a + 3$
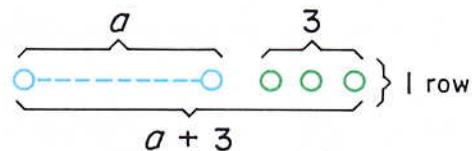
You must have drawn $a + 3$ circles. Now think of a number (b) and draw that number of rows like the one above.

a 3

b rows { ○------○ ○ ○ ○ } b rows
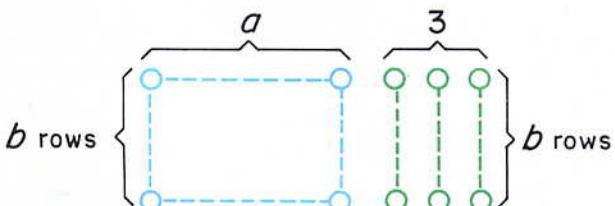
There are $a \times b$ or ab blue circles. There are $3 \times b$ or $3b$ green circles. There are $ab + 3b$ circles altogether.

Or, we can say that altogether there are $b(a + 3)$ circles.

Each diagram below helps indicate a certain idea. Complete the sketch and the shorthand that suggests the same idea.

n

●------●
●------●
●------● } 4
●------●

$4 \times n$ or ____

x ---

●------● ● ● ● ● ●

$x + $ ____

____

y

□------□
| |
□------□ } y

$y \cdot y$ or ____

m n

□------□ ○------○

____

n 2

●------● | |
| | | |
●------● | | } n

$n^2 + $ ____

or

$n(\ + \ )$ ____

---   ---

● ● ● ● ● ●
| |
● ● ● ● ● ● } x

$3x + $ ____

$x(\ + \ )$ ____

or $7x$ ____

s

●------●
| |
●------● } t ____

121
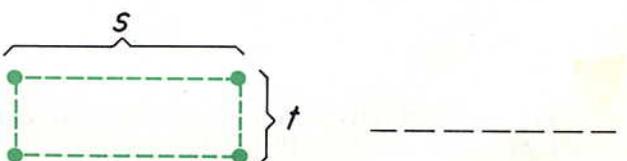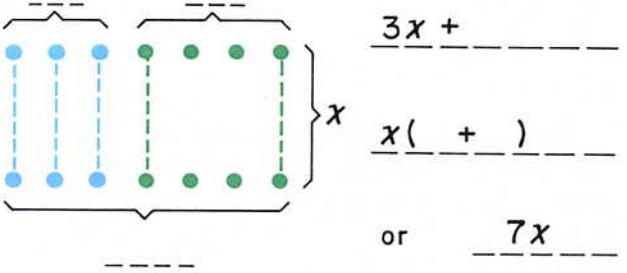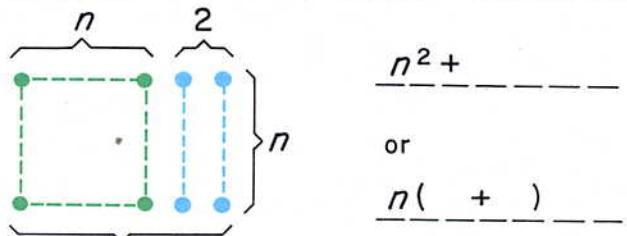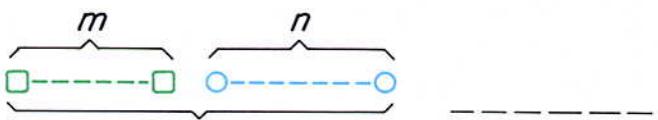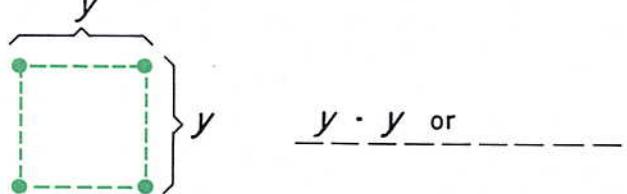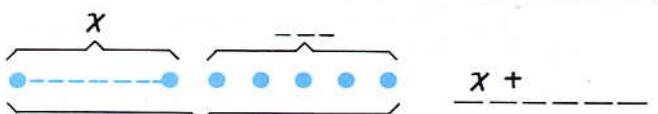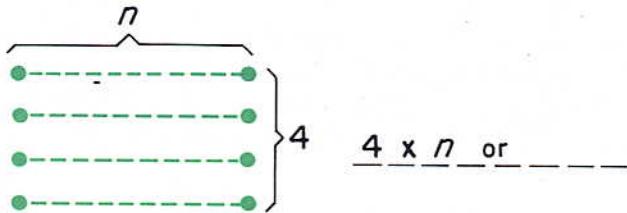
Jane said to Al, Jack, Betty, Dick, and Ruth, "Each of you think of a number. Draw a row with that number of dots. Now add 2 more dots to the row.

"Think again of the number you first thought of and draw that many rows just like your first row. I don't know what you've drawn, but a sketch helps me see clearly what I do know.
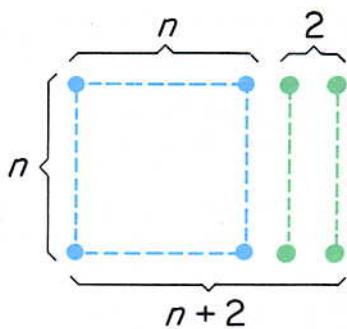
"I use an n to remind me of whatever number you thought of. So, I can say that you have drawn $n^2 + 2n$ dots. Or, I can say that you have drawn $n(n + 2)$ dots.

"Each of you fill in this chart and see if I am not right.

		Al	Jack	Betty	Dick	Ruth
Your number	n	5	8			20
	n^2	25			49	
	$2n$	10		22		
	$n^2 + 2n$					
	n	5				
	$n + 2$					
	$n(n + 2)$					

"No matter what whole number you think of," Jane explained, "my sketch remains the same. And it suggests that, for each whole number n,

$$n(n + 2) = n^2 + 2n$$

"Try it with any whole number you can think of. The results will be the same."

Jane asked her friends to try another experiment with her.

"Think of another whole number. I'm going to use an x to remind me of it. Draw that many dots in the top row. Then add 4 more.

"Next, draw as many rows like your first row as the number you first thought of.

"My sketch looks much like the earlier one. But, now add 2 more rows. There is my sketch:
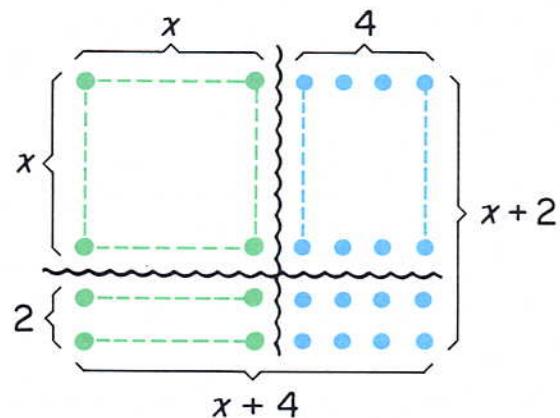
"Another diagram leads me to the conclusion that, for each whole number x,

$$(x + 2)(x + 4) = x^2 + 6x + 8$$

"Test my conclusions in the chart below."

	Al	Jack	Betty	Dick
x	6			
$x + 2$		11		
$x + 4$				
$(x+2)(x+4)$			35	
x^2				25
$6x$				
$x^2 + 6x + 8$				

Make your own chart like the one above. Think of other whole numbers. Keep experimenting until you are completely confident that there will be no surprises.

Drawing dots gets tiresome. Another kind of sketch might look like this:

$$(x+3)(x+5) = x^2 + \underline{\quad} x + \underline{\quad}$$

$x + 7$

$$xy + \underline{\quad\quad} = y(\underline{\quad} + \underline{\quad})$$

$b + 15$
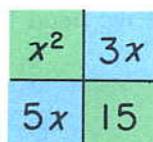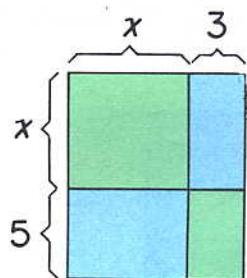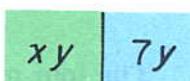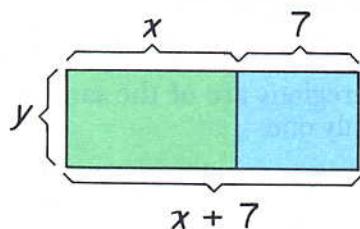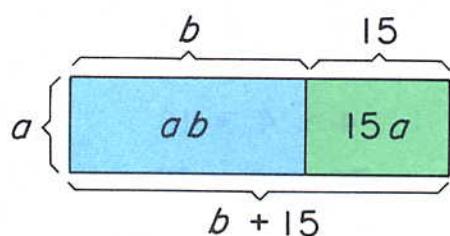
$$ab + \underline{\quad} = a(\underline{\quad} + \underline{\quad})$$
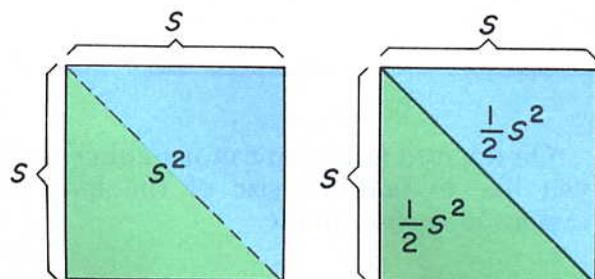
$$\tfrac{1}{2}s^2 + \tfrac{1}{2}\underline{\quad} = \underline{\quad}$$

The blue and green regions together suggest s^2. The green region suggests $\tfrac{1}{2}s^2$. The blue region also suggests $\tfrac{1}{2}s^2$.
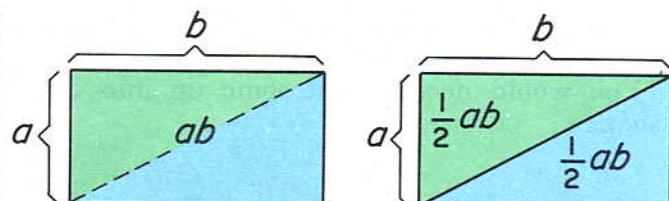
$$\tfrac{1}{2}\underline{\quad} + \tfrac{1}{2}\underline{\quad} = \underline{\quad}$$

a and b remind us of two numbers. If we had known that those numbers were 2 and 4, we could have drawn the following sketch:
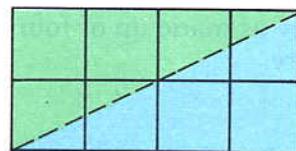
It would take $\tfrac{1}{2}$ of 2×4 small squares to cover the green region — 4 squares; and $\tfrac{1}{2}$ of 2×4 to cover the blue region. This is a special example of the idea of the sketch at the top of this column.
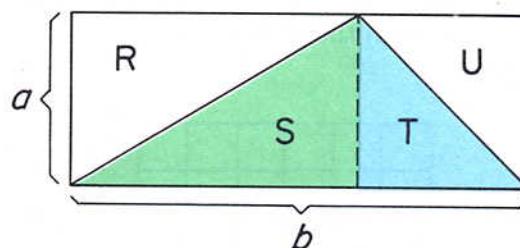
The large rectangle is a sketch that suggests ab.

Is the region labeled R the same size as the green region labeled S? _____

Is region U the same size as the blue region T? _____

Are regions R and U together the same size as regions S and T together? _____

Can we say that the colored region (blue and green together) suggests $\tfrac{1}{2}ab$? _____

123

How many small squares ☐ would you need to cover the large blue square below? (You would need to cut some up into odd shapes.)
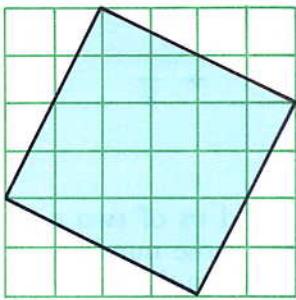
One way to think about the question is to see the square as made up of four triangles and a small square.
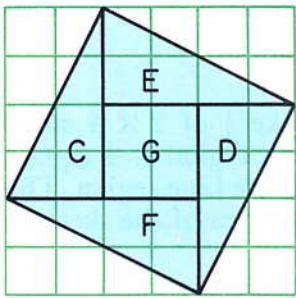
Each blue region is labeled so we can talk about it.

These four triangles and the square can be rearranged as in the following diagram:
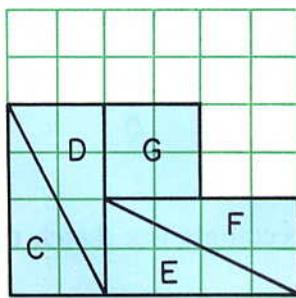
Now you can easily count the number of small squares you would need to cover the blue region.

Here is another approach to the same problem. In the entire diagram, there are 6 × 6, or 36 small squares. If we can find the number of squares we would need to cover the green regions below, we can get to our result. We would simply subtract the number of whole squares that are green from 36. That will tell us all we need to know about the blue squares.
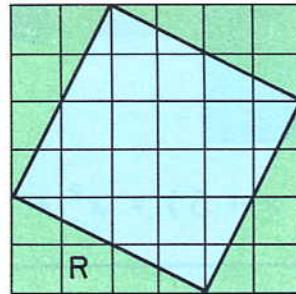
The four green regions are of the same size. So, let's consider only one.
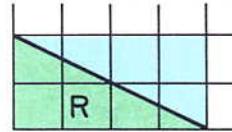
The blue and green shaded regions exactly cover eight squares. So it must take four square pieces to cover R.

$$4 \times 4 = 16$$
$$36 - 16 = 20$$

So again we have found the answer to our question. We would be surprised if we arrived at different answers.

On the next page you can use either method you like to find the size of the large blue regions in each example.

After you have completed most of the chart at the bottom of the next page, look for an important pattern.

In each column, how can you know the result that will appear in the bottom row from the two results above it?

Complete the record below. Indicate the length of green lines in units of ●—●. Record the number of small squares required to cover each large blue square.
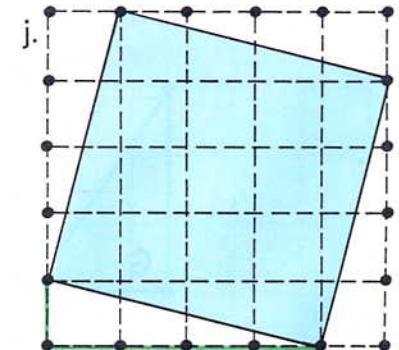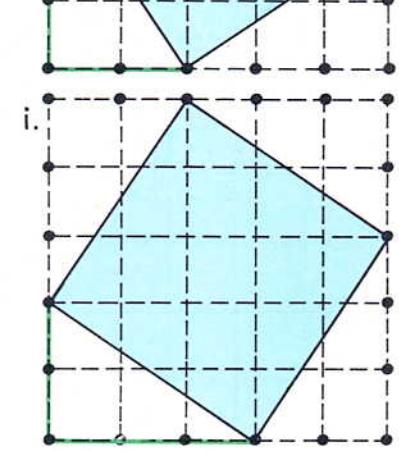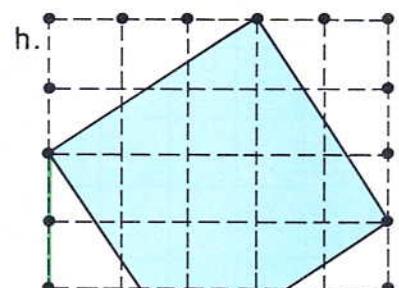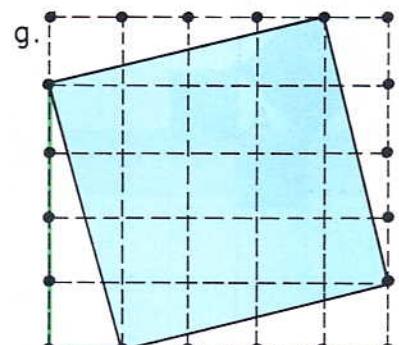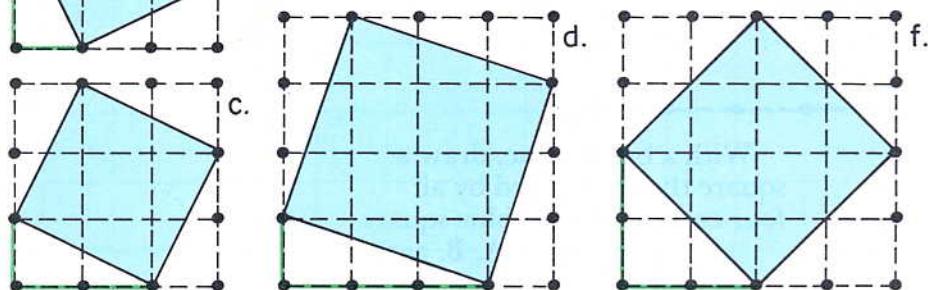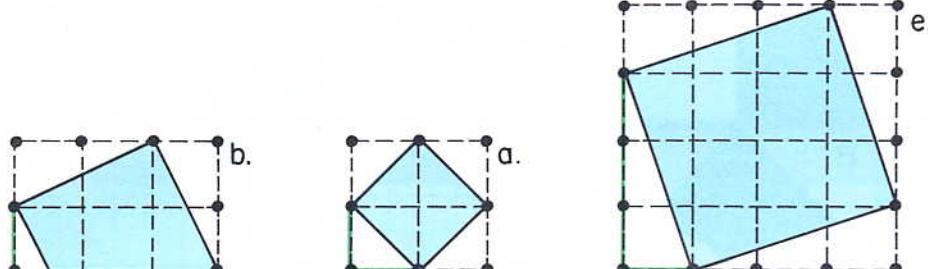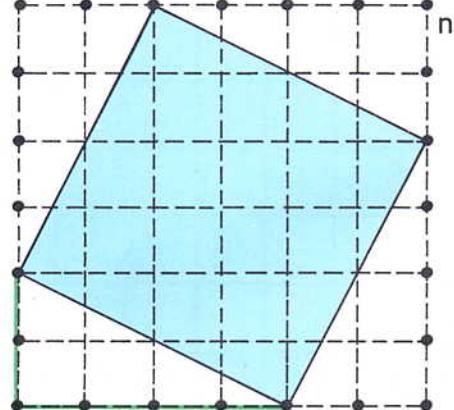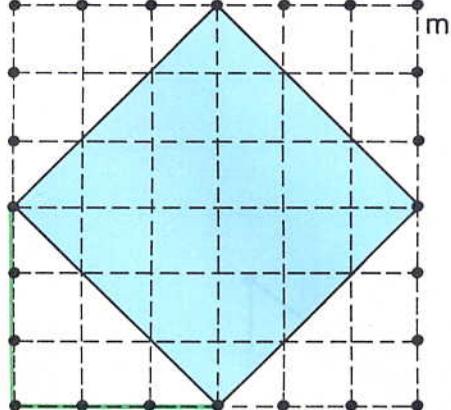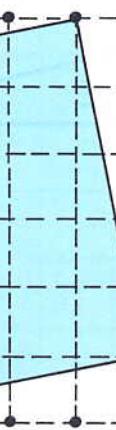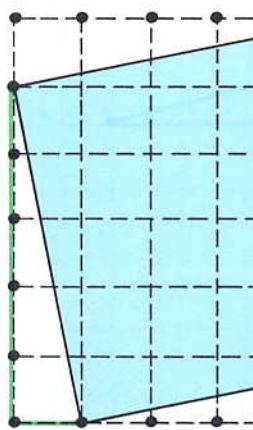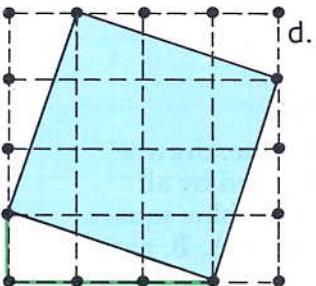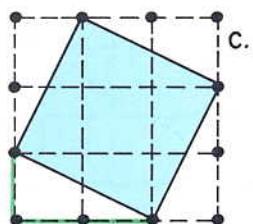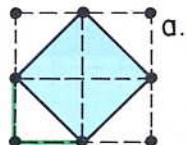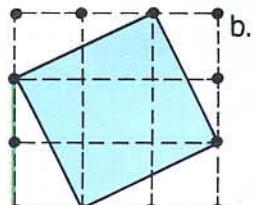

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	x	y	z
No. of ⌊●	1	2													6	7	3
No. of ●—●	1	1													8	2	4
Squares shaded blue	2	5															

Draw blue squares with blue lines as sides. Draw green squares with green lines.
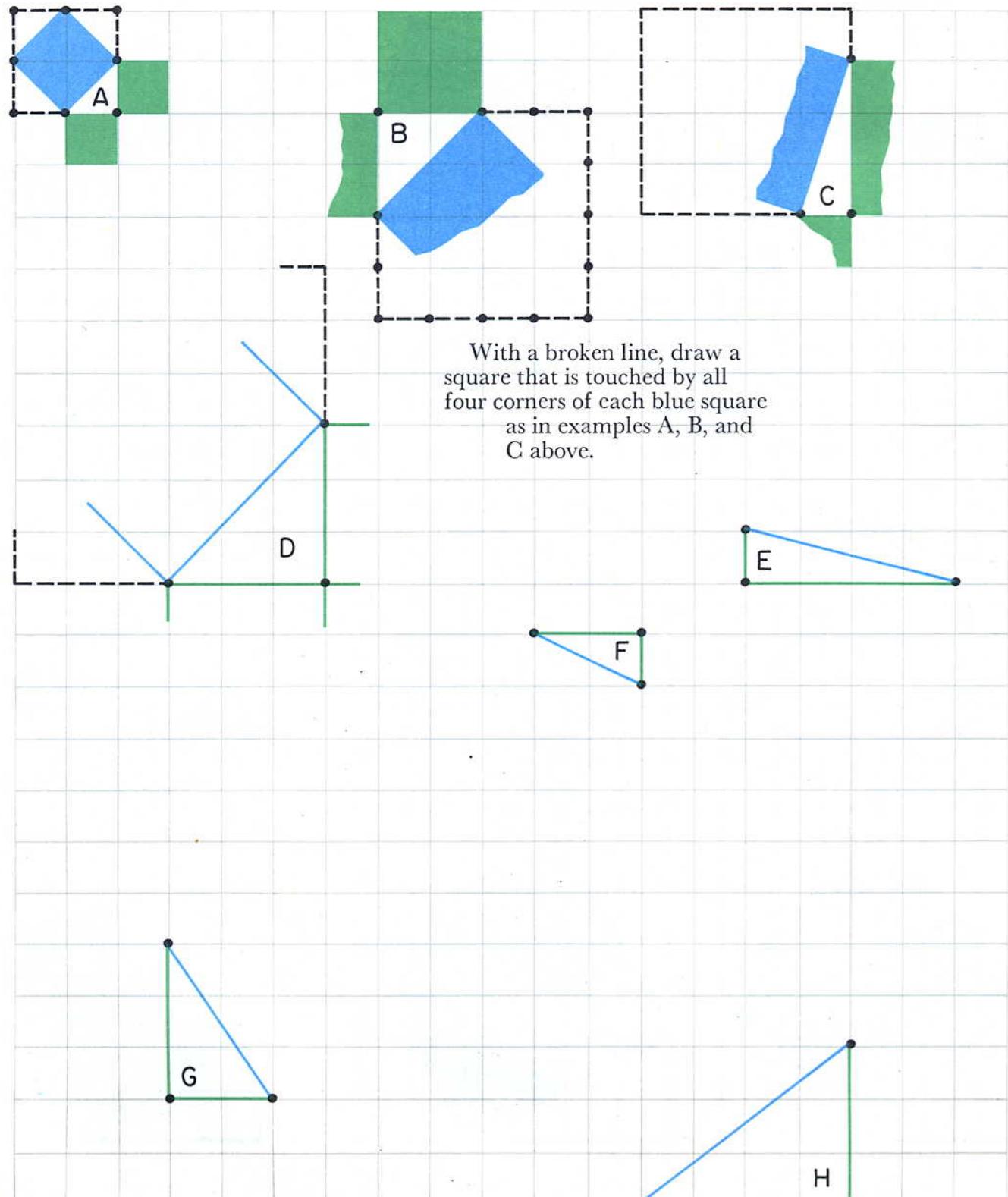
With a broken line, draw a
square that is touched by all
four corners of each blue square
as in examples A, B, and
C above.

	A	B	C	D	E	F	G	H	V	W
No. top or bottom	1	4							36	
No. on side	1									25
No. in blue square	2	8							52	50

Let's do some SPACE MATHEMATICS.

Before we start, you must realize that we shall, of course, run into some very large distances — and some very large numbers.

Col. John Glenn is the first American astronaut to orbit the earth. Some of the time, he was 100 miles above the earth. How far do you think he could see in all directions?

Let's see if we can find out!

Here is a sketch that will help us find out (our sketch is not to scale).

Col. John Glenn's Mercury capsule
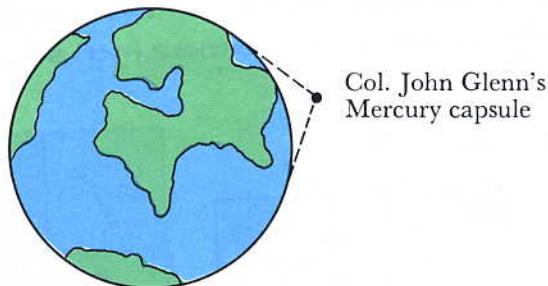
If we took a slice right through the center of the earth, we would see the problem this way:
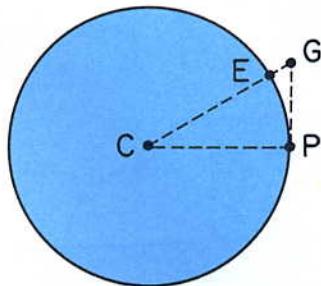
G suggests Glenn, and from point E he is 100 miles above the earth. From that height, he can see as far as point P on the surface of the earth. Of course, point E and point P are about the same distance from C — the center of the earth — and it is about 4,000 miles from any point on the earth to the center of the earth.

Let's draw a larger sketch of that part of our diagram we are most interested in and fill in some distances.
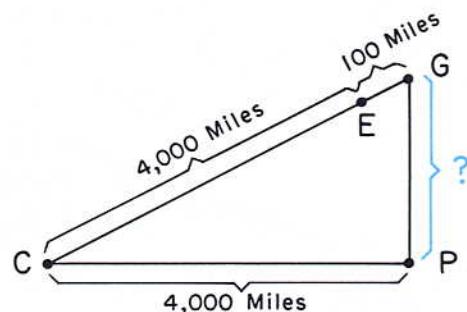
How far is Glenn from the center of the earth? (4,000 + 100 = _____.)

He can see from G to P. How far is that?
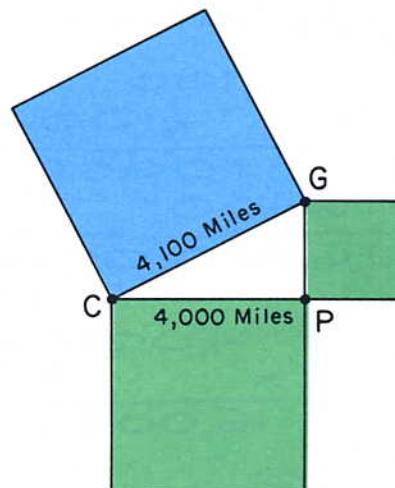
If the blue square is 4,100 miles on each side, how many square miles are there in the blue square? _____

How many square miles in the large green square? _____

How many square miles in the small green square? _____

How long is each side of the small green square? _____

That length is the distance Col. Glenn can see in every direction. (Find out yourself before turning the page.)

127
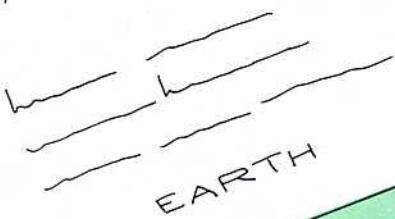
Dear Friends,

I'm sorry I had to leave without saying goodbye. On my way home, I worked out the answer to that problem about how far Col. Glenn could see from 100 miles out in space. Here is my computation:

$$
\begin{array}{r}
4,100 \\
\times\quad 4,100 \\
\hline
410000 \\
16400 \\
\hline
16,810,000
\end{array}
$$

$$
\begin{array}{r}
4,000 \\
\times\quad 4,000 \\
\hline
16,000,000
\end{array}
$$

$$
\begin{array}{r}
16,810,000 \\
-16,000,000 \\
\hline
810,000
\end{array}
$$

$$810,000 = 900 \times 900$$

So, Col. John Glenn could see about 900 miles on Earth in every direction.

This means he could look over about 2,500,000 square miles of Earth scenery.

Your friend,

Alec Marson

P.S. Of course, I could have found the distance Col. Glenn could see by a much easier method that I learned in Algebra. I learned that, for any numbers (I use x and y here, as reminders), it is true that

$$x^2 - y^2 = (x+y)\cdot(x-y)$$
$$4,100^2 - 4,000^2 = (8,100)\cdot(100)$$
$$= 810,000$$
$$= 900 \times 900$$

Here is a series of sketches I use to help me remember and understand that amazing short-cut.
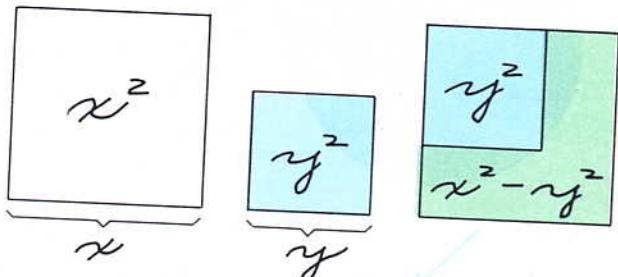
The green region suggests the idea of $x^2 - y^2$. How big is it?
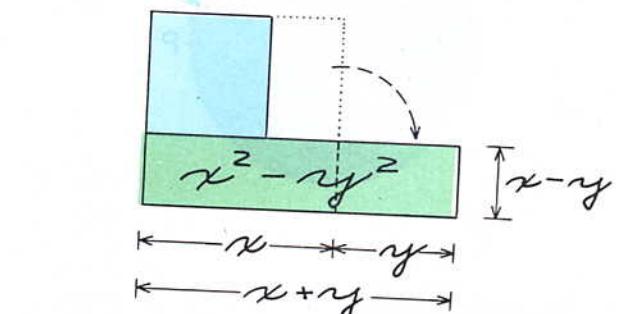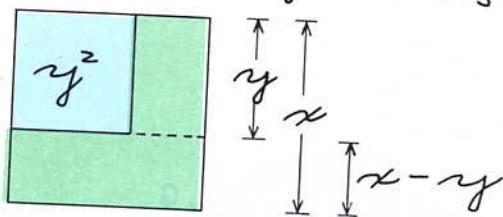
The green region certainly suggests the idea of $(x+y)\cdot(x-y)$. Clearly $x^2 - y^2 = (x+y)\cdot(x-y)$.